30. On Multi-valued Monotone Closed Mappings

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V. I. Ponomaleff [1] has defined the new space κX for T_1-space X. According to him the space κX is the set of all non-empty closed subsets of X and topology is defined as follows: for each point (F_0) of κX and for every neighborhood OF_0 of F_0 in X $D_1(OF_0)$ is the set of all closed subsets of X contained in OF_0 and these $D_1(OF_0)$ form the bases of the neighborhoods of (F_0) in κX. In our paper we shall use his definition for the topological space X (without T_1-axiom).

A multi-valued mapping f of a topological space X into a topological space Y is *monotone* if for each point x of X fx is closed in Y and for each pair of distinct points x and x' of X $fx\frown fx'=\phi$.

We use the definitions due to him: the continuity of a mapping f of X into Y is that for every point x of X and for each neighborhood Ofx of fx in Y there is a neighborhood Ox of x in X such that $fOx\subset Ofx$; the closedness of f is the closedness of the image of every closed subset of X; $\bar{f}$ is a one-valued mapping of X into κY which maps every point x of X to a point (fx) of κY.

Theorem 1. *If f is a one-valued closed continuous mapping of a topological space X onto a T_1-space Y, then the inverse mapping f^{-1} is a multi-valued monotone closed continuous mapping of Y onto X. Conversely, if g is a multi-valued monotone closed continuous mapping of a topological space X onto a topological space Y and if for every point y of Y $g^{-1}(y)=x$ such that $gx\ni y$, then g^{-1} is a one-valued closed continuous mapping of Y onto X.*

Proof. Since f is continuous, f^{-1} is closed, and since Y is T_1-space, f^{-1} is monotone. To prove that f^{-1} is continuous, let y be an arbitrary point of Y and $Of^{-1}(y)$ be an arbitrary neighborhood of $f^{-1}(y)$ in X. Since f is closed, there is an open inverse set $(Of^{-1}(y))_0$*) such that $f^{-1}(y)\subset(Of^{-1}(y))_0\subset Of^{-1}(y)$. Then $V=f(Of^{-1}(y))_0$ is a neighborhood of y in Y such that $f^{-1}(V)=(Of^{-1}(y))_0\subset Of^{-1}(y)$. This completes the proof that f^{-1} is a multi-valued monotone closed continuous mapping.

Conversely, let g be a multi-valued monotone closed continuous mapping of X onto Y. To show that g^{-1} is closed, let A be an arbitrary closed subset of Y. Since $g^{-1}(A)=\{x|gx\frown A\neq\phi;\ x\in X\}$, and if x_0 is an arbitrary point of $X-g^{-1}(A)$, then $gx_0\frown A=\phi$; that is, $gx_0\subset X-A$.

*) $(Of^{-1}(y))_0$ is the union of all $f^{-1}(p)$ $(p\in Y)$ such that $f^{-1}(p)\subset Of^{-1}(y)$.

Since g is continuous, there is a neighborhood Ox_0 of x_0 in X such that $gOx_0 \subset X-A$. This shows $gOx_0 \frown A=\phi$ and $Ox_0 \subset X-g^{-1}(A)$. So $X-g^{-1}(A)$ is open and $g^{-1}(A)$ is closed. Finally, we shall prove that g^{-1} is continuous. Let y_0 be an arbitrary point of Y and let $Og^{-1}(y_0)$ be an arbitrary neighborhood of $g^{-1}(y_0)$ in X. Since g is closed, $g(X-Og^{-1}(y_0))$ is closed in Y and since $g(X-Og^{-1}(y_0))=\bigcup\{gx|x\notin Og^{-1}(y_0); x\in X\}=\bigcup\{gx|gx\frown gOg^{-1}(y_0)=\phi;\ x\in X\}$, $g(X-Og^{-1}(y_0))\not\ni y_0$. Let $U=Y-g(X-Og^{-1}(y_0))$, then U is a neighborhood of y_0 in Y and $g^{-1}(U)=\{x|gx\frown U\neq\phi;\ x\in X\}=\{x|gx\frown g(X-Og^{-1}(y_0))=\phi;\ x\in X\}=\{x|x\in Og^{-1}(y_0); x\in X\}=Og^{-1}(y_0)$; that is, g^{-1} is continuous at y_0. Then g^{-1} is continuous and this completes the proof.

In the following, we shall prove the invariance of topological properties under a multi-valued monotone closed continuous mapping under some restrictions.

Lemma 1. *If f is a multi-valued monotone closed continuous mapping of a topological space X into a topological space Y, then $\bar{f}$ is a (one-valued) closed continuous mapping of X onto $\bar{f}X$ (in κY).*

Proof. The continuity of $\bar{f}$ is followed from [1]. We shall prove the closedness of $\bar{f}$. Let F be an arbitrary closed subset of X, then $\bar{f}F=\{(fx)|x\in F\}$, so it is sufficient to prove that $\bar{f}X-\bar{f}F$ is open in $\bar{f}X$. Let (fx_0) be an arbitrary point of $\bar{f}X-\bar{f}F$, then $x_0\notin F$; that is, $fx_0\frown fF=\phi$. By the closedness of f $V=Y-fF$ is an open subset of Y containing fx_0 and so $D_1(V)\frown\bar{f}X$ is an open subset of $\bar{f}X$ containing (fx_0). Since $V=Y-fF$, $D_1(V)\frown\bar{f}F=\phi$. This shows that $\bar{f}X-\bar{f}F$ is open. Thus Lemma 1 is proved.

Lemma 2. *Let f be a multi-valued monotone closed continuous mapping of a topological space X onto a topological space Y. If $D_1(U)$ is a non-empty open subset of κY, then $\tilde{U}=\bigcup_{(fx)\in D_1(U)} fx$ is open in Y.*

Proof. By the continuity of $\bar{f}$, $V=\bar{f}^{-1}D_1(U)=\{x|(fx)\in D_1(U)\}$ is open in X. Since f is closed, $f(X-V)$ is closed in Y. But $f(X-V)=\bigcup_{x\notin V} fx=Y-\bigcup_{x\in V} fx=Y-\bigcup_{(fx)\in D_1(U)} fx$, so $\bigcup_{(fx)\in D_1(U)} fx=\tilde{U}$ is open in Y. This completes the proof.

Lemma 3. *Let X be a normal space and A be a closed subset of X. If $\{U_i\}$ is a countable star-finite open covering of A, then there is a countable locally finite collection $\{V_i\}$ of open subsets of X such that $V_i\frown A\subset U_i$ $(i=1,2,\cdots)$ and $\{V_i\frown A\}$ covers A.*

Proof. We shall prove it by induction. For U_1, let $F_1=A-\bigcup_{i\neq 1} U_i$ and $F_1'=A-U_1$, then they are disjoint closed subsets of X. (If $F_1=\phi$, we can omit U_1 from $\{U_i\}$ and if $F_1'=\phi$, we begin from U_2.) Since X is normal, there is an open subset G_1 of X such that $F_1\subset G_1$ and

$\bar{G}_1 \frown F_1' = \phi$. Then $G_1 \frown A \subset U_1$ and $\{G_1 \frown A, U_2, \cdots\}$ is an open covering of A. If $U_1 \frown U_i = \phi$ for some i $(i>1)$, then $U_i \subset A - U_1$ and $A - U_1$ is closed, so $\bar{G}_1 \frown \bar{U}_i = \phi$. We assume that there is a collection $\{G_i | i<n\}$ of open subsets of X such that $\{G_1 \frown A, G_2 \frown A, \cdots, G_{n-1} \frown A, U_n, U_{n+1}, \cdots\}$ is an open covering of A and $G_i \frown A \subset U_i$ $(i<n)$. Moreover, for some i $(i<n)$ $U_i \frown U_j = \phi$ implies $\bar{G}_i \frown \bar{G}_j = \phi$ if $j<n$, and $\bar{G}_i \frown \bar{U}_j = \phi$ if $j \geqq n$. Now we shall construct G_n satisfying the above conditions. Let $F_n = A - \{(\smile_{i<n} G_i) \smile (\smile_{i>n} U_i)\}$ and $F_n' = (A - U_n) \smile \bigcup_i \{\bar{G}_i | U_i \frown U_n = \phi;\ i<n\}$.

Then F_n and F_n' are closed subsets of X and are disjoint by the assumption of induction. Since X is normal, there is an open subset G_n of X such that $F_n \subset G_n$ and $\bar{G}_n \frown F_n' = \phi$. Then $G_n \frown A \subset U_n$ and $\{G_1 \frown A, \cdots, G_n \frown A, U_{n+1}, \cdots\}$ is an open covering of A. Let $U_i \frown U_n = \phi$ for some i. If $i<n$, then $\bar{G}_i \subset F_n'$; so $\bar{G}_i \frown \bar{G}_n = \phi$. If $i>n$, $U_i \subset A - U_n$; so $\bar{U}_i \subset A - U_n \subset F_n'$; that is, $\bar{U}_i \frown \bar{G}_n = \phi$. By induction we have a collection $\{G_i\}$ of open subsets of X. First we show that $\{G_i \frown A\}$ is a covering of A. Let x be an arbitrary point of A. Since $\{U_i\}$ is star-finite, there is a number n_0 such that $j>n_0$ implies $x \in U_j$. Then from the covering $\{G_1 \frown A, \cdots, G_{n_0} \frown A, U_{n_0+1}, \cdots\}$ of A x is contained in some $G_i \frown A$ $(i \leqq n_0)$. This shows that $\{G_i\}$ covers A. Next, if $U_i \frown U_j = \phi$ $(i>j)$, by the induction the collection $\{G_1, \cdots, G_i, U_{i+1}, \cdots\}$ shows $\bar{G}_i \frown \bar{G}_j = \phi$. Finally, we construct the desired $\{V_i\}$. If $\{G_i\}$ is locally finite in X, let $G_i = V_i$. If $\{G_i\}$ is not locally finite in X, let X_0 be the set of all points at which $\{G_i\}$ is not locally finite. Then X_0 is closed in X and $X_0 \frown A = \phi$. Indeed, the closedness of X_0 is easy from the open of $X - X_0$. Let x be an arbitrary point of A. Since $\{U_i\}$ is star-finite, only finite number $\{U_{i_0}, \cdots, U_{i_n}\}$ of $\{U_i\}$ contain x and there is a number n_0 such that $k>n_0$ implies $U_k \frown U_{i_j} = \phi$ for each j, $j \leqq n$. This shows that if we let the neighborhood of x be $Ox = \bigcap_i \{G_i | G_i \ni x\}$, then Ox intersects only G_i $(i \leqq n_0)$; that is, $\{G_i\}$ is locally finite at x. Now we have $X_0 \frown A = \phi$. Since X is normal, there is an open subset U of X such that $A \subset U$ and $\bar{U} \frown X_0 = \phi$. If we let $V_i = G_i \frown U$, then $\{V_i\}$ is locally finite collection of open subsets of X which cover A and $V_i \frown A \subset G_i \frown A \subset U_i$.

Theorem 2. *Let f be a multi-valued monotone closed continuous mapping of a topological space X onto a topological space Y and let for every point x of X fx be an S-space*[**] *with Lindelöf property. If X is paracompact and Y is normal, then Y is paracompact.*

Proof. By Lemma 1 $\bar{f}$ is a closed continuous mapping of X onto $\bar{f}X$ and by the definition $\bar{f}$ is one-to-one, $\bar{f}X$ is paracompact. Let

**) A space is S-space if every open covering has the star-finite open covering as refinement.

$\mathfrak{U}=\{U_\alpha\}$ be an arbitrary open covering of Y. Since for every point x of X fx is an S-space with the Lindelöf property, by Lemma 3 there is a countable locally finite collection $\{V^x_{\alpha_i} \mid i=1,2,\cdots\}$ of open subsets of Y such that $V^x_{\alpha_i} \frown A$ contained in some element of $\mathfrak{U}$. Let $V^x=\smile_i V^x_{\alpha_i}$. Then $\{D_1(V^x)\frown \bar{f}X \mid x\in X\}$ is an open covering of $\bar{f}X$. Since $\bar{f}X$ is paracompact, there is a locally finite open covering $\{D_1(V_\beta)\frown \bar{f}X \mid \beta\in\Omega\}$ which is a refinement of $\{D_1(V^x)\frown\bar{f}X \mid x\in X\}$. By Lemma 2 for each $\beta\in\Omega$ there is an open subset $\widetilde{V}_\beta$ of Y such that $D_1(\widetilde{V}_\beta)\frown\bar{f}X=D_1(V_\beta)\frown\bar{f}X$ and $\widetilde{V}_\beta\subset V_\beta$. For each $\beta\in\Omega$ we pick up one V^x such that $D_1(\widetilde{V}_\beta)\frown\bar{f}X \subset D_1(V^x)\frown\bar{f}X$ and let $W^\beta_{\alpha_i}=V^x_{\alpha_i}\frown\widetilde{V}_\beta$ $(i=1,2,\cdots)$. Then $\{W^\beta_{\alpha_i} \mid i=1,2,\cdots;\ \beta\in\Omega\}$ is a locally finite open covering of Y. Indeed, since $W^\beta_{\alpha_i}$ is open and $\{\widetilde{V}_\beta \mid \beta\in\Omega\}$ covers Y, $\{W^\beta_{\alpha_i} \mid i=1,2,\cdots;\ \beta\in\Omega\}$ is an open covering of Y. Let y be an arbitrary point of Y. Then there is a point x of X such that $y\in fx$. Since $\{D_1(\widetilde{V}_\beta)\frown\bar{f}X \mid \beta\in\Omega\}$ is locally finite, the only finite number $\{D_1(\widetilde{V}_{\beta_i})\frown\bar{f}X \mid i=1,2,\cdots,n\}$ intersect the neighborhood $D_1(Ofx)$ of (fx); that is, $\widetilde{O}fx$ intersects only $\widetilde{V}_{\beta_i}$ $(i=1,2,\cdots,n)$. Since for each i $\widetilde{V}_{\beta_i}=\bigcup_{j=1}^{\infty} W^{\beta_i}_{\alpha_j}$ and $\{W^{\beta_i}_{\alpha_j} \mid j=1,2,\cdots\}$ is locally finite, there is a neighborhood O_iy of y which intersects only finite number of $\{W^{\beta_i}_{\alpha_j} \mid j=1,2,\cdots\}$. Let $Oy=\widetilde{O}fx\frown\bigcap_{i=1}^{n}O_iy$. Then Oy is a neighborhood of y and intersects only finite number of $\{W^\beta_{\alpha_i} \mid i=1,2,\cdots;\ \beta\in\Omega\}$. This shows that $\{W^\beta_{\alpha_i} \mid i=1,2,\cdots;\ \beta\in\Omega\}$ is locally finite. Moreover, each $W^\beta_{\alpha_i}\subset$ some $V^x_\alpha\subset$ some $U\in\mathfrak{U}$ shows that $\{W^\beta_\alpha \mid i=1,2,\cdots;\ \beta\in\Omega\}$ is a refinement of $\{U_\alpha\}$. This completes the proof.

If we use the Lemma 1 of [2], we have the following theorem in the same way:

Theorem 3. *Let f be a multi-valued monotone closed continuous mapping of a topological space X onto a topological space Y and let for every point x of X fx be paracompact (countably paracompact). If X is paracompact and Y is collectionwise normal, then Y is paracompact (countably paracompact).*

We know that, for a (one-valued) closed continuous mapping, if the inverse image of every point is compact, then the paracompactness (countably paracompactness) is invariant under the inverse mapping (see [3]).

By Theorems 1 and 2 (or 3) we have that, for a (one-valued) closed continuous mapping f of a topological space X onto a T_1-space Y, if X is normal (collectionwise normal) and for every point y of Y $f^{-1}(y)$ is an S-space with the Lindelöf property (paracompact), then the paracompactness is invariant under the inverse mapping of f. In particular, if $f^{-1}(y)$ is countably paracompact in collectionwise normal space X, then X is countably paracompact.

References

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