25. A Characterization of Real Analytic Functions

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1. Introduction. It is well known that a C^∞ function f is analytic on $[\alpha, \beta]$ if and only if there exist positive constants M and a such that

$$(1)\qquad \sup_{x\in[\alpha,\beta]} |f^{(k)}(x)| \leqq M a^k k!, \quad k=0,1,2,\cdots .$$

In this paper we prove a generalization of this fact for functions with several variables. Our main result is the following

Theorem. Let D be a domain in R^n, and let A be an elliptic differential operator of order m with constant coefficients. Then, for a function $f\in L^2_{\mathrm{loc}}(D)$ to be analytic in D it is necessary and sufficient that 1) for every k, $A^k f$ (in the sense of the distribution) belongs to $L^2_{\mathrm{loc}}(D)$, and that 2) for every compact $K\subset D$, there exist positive constants M and a such that*)

$$(2)\qquad \| A^k f\|_K \leqq M(ak)^{mk}, \quad k=0,1,2,\cdots .$$

Recently E. Nelson gave a similar sufficient condition in the case where the coefficients of A are analytic [5]. His condition is essentially that

$$(3)\qquad \| A^k f\|_K \leqq M(ak)^{k}, \quad k=0,1,2,\cdots .$$

It is highly desirable to obtain a result which includes the above two cases.

At the end of this paper an application will be given on the regularity of solutions of parabolic differential equations.

Here the author wishes to express his cordial thanks to Professor K. Yosida whose instruction has meant much to him.

2. Proof of the theorem. We prepare several lemmas. Lemma 1 can be proved by using Cauchy's integral formula and Taylor expansion.

Lemma 1. Let K be a compact convex set in R^n. A C^∞ function $f(x)$ defined on K is analytic if and only if there exist positive constants M and a satisfying

$$(4)\qquad \sup_{x\in K} |D^p f(x)| \leqq M a^{|p|} p!, \quad |p|=0,1,2,\cdots,$$

where
$$p=(p_1,\cdots,p_n),\quad |p|=p_1+\cdots+p_n,$$
$$D^p=\partial^{|p|}/\partial x_1^{p_1}\cdots\partial x_n^{p_n},\quad p!=p_1!\,p_2!\cdots p_n!.$$

*) $\|f\|_K=\left(\int_K |f(x)|^2 dx\right)^{1/2}$. But the theorem holds for norms other than the L^2-norm, too. For some system of differential operators an analogous theorem holds, of which Proposition 1 is a special case.

If a function f is defined in D and if its derivatives in the sense of the distribution up to order k are all square integrable, then we call f k times strongly differentiable and write $f\in H^k(D)$. Let

$$|||f|||_{j,D}=(\sum_{|p|=j}\|D^pf\|^2_D)^{1/2}. \tag{5}$$

Then $H^k(D)$ forms a Hilbert space by the norm

$$\|f\|_{k,D}=\Big(\sum_{j=0}^{k}|||f|||^2_{j,D}\Big)^{1/2}.$$

Lemma 2 (Soboleff). Let D be a domain in R^n. If a function $f\in H^k(D)$ and $h=k-[(n+1)/2]$, then f belongs to $C^h(D)$ and

$$\sup_{x\in D_\rho}|D^pf(x)|\leqq C\sum_{j=|p|}^{|p|+[(n+1)/2]}|||f|||_j\rho^{j-\frac{n}{2}-|p|} \quad \text{for} \quad |p|\leqq h, \tag{6}$$

where D_ρ is the set $\{x\in D; |y-x|<\rho\Rightarrow y\in D\}$, and constant C depends only on n.

As for proof see [1] or [6], but a simplest proof is obtained by the techniques of the Fourier transformation.

Combining the above two lemmas we have

Proposition 1. Let f be a funntion defined in D. Then f is analytic in D if and only if $f\in H^k_{\text{loc}}(D)$ for every k, and for every compact K, there exist positive constants M and a such that

$$|||f|||_{k,K}\leqq M(ak)^k, \quad k=0,1,2,\cdots. \tag{7}$$

For a function f in $L^2_{\text{loc}}(D)$ we shall use the notation

$$N_{j,\rho,D}(f)=\sup_{0<\delta\leqq\rho}\delta^j\|f\|_{D_\delta}. \tag{8}$$

Especially

$$N_{0,\rho,D}(f)=N_{0,D}(f)=\|f\|_D.$$

Lemma 3. Let Ω be a bounded domain in R^n, and A be an elliptic differential operator of order m with constant coefficients. Suppose $f\in L^2_{\text{loc}}(\Omega)$ and $Af\in L^2_{\text{loc}}(\Omega)$. Then $D^pf\in L^2_{\text{loc}}(\Omega)$ for $|p|\leqq m$, and the following inequality holds.

$$N_{|p|,\rho,\Omega}(D^pf)\leqq C(N_{m,\rho,\Omega}(Af)+N_{0,\Omega}(f))^{|p|/m}(N_{0,\Omega}(f))^{1-|p|/m}, \tag{9}$$

for $0<\rho\leqq\rho_0$, where constant C depends only on n, ρ_0, and A.

We can prove this by the same method as in Hörmander [2].

Corollary. Under the same conditions as in Lemma 3 the following inequality holds.

$$\|D^pf\|_{\Omega_{\rho+\sigma}}\leqq C\|Af\|^{|p|/m}_{\Omega_\sigma}\|f\|^{1-|p|/m}_{\Omega_\sigma}+C\frac{1}{\rho^{|p|}}\|f\|_{\Omega_\sigma}. \tag{10}$$

Proof. We have, from Lemma 3 and (8),

$$\begin{aligned}\rho^{|p|}\|D^pf\|_{\Omega_{\rho+\sigma}}&\leqq N_{|p|,\rho,\Omega_\sigma}(D^pf)\\ &\leqq C(N_{m,\rho,\Omega_\sigma}(Af)+N_{0,\Omega_\sigma}(f))^{|p|/m}(N_{0,\Omega_\sigma}(f))^{1-|p|/m}\\ &\leqq C(\rho^m\|Af\|_{\Omega_\sigma}+\|f\|_{\Omega_\sigma})^{|p|/m}(\|f\|_{\Omega_\sigma})^{1-|p|/m}\\ &\leqq 2C((\rho^m\|Af\|_{\Omega_\sigma})^{|p|/m}+\|f\|^{|p|/m}_{\Omega_\sigma})(\|f\|_{\Omega_\sigma})^{1-|p|/m}\end{aligned} \tag{11}$$

and, dividing by $\rho^{|p|}$, we have (10).

Proof of the theorem. Necessity is easily proved from Lemma 1.

For every point $x \in D$, we can choose a sphere Ω with center x which is contained entirely in D. It is sufficient to prove that f is analytic in $\Omega_{2\rho}$, where 2ρ is smaller than the radius of Ω.

M in (2) may be replaced by 1 for sufficiently large a. We have, by (10),

$$
\begin{aligned}
|||f|||_{km,\Omega_\rho} &\leqq \sum_{|p|=km} \|D^p f\|_{\Omega_\rho} = \sum_{|p|=m} \sum_{|q|=(k-1)m} \|D^p D^q f\|_{\Omega_\rho} \\
&\leqq C_1 \sum_{|q|=(k-1)m} \|AD^q f\|_{\Omega_{\rho(1-1/k)}} + C_2\left(\frac{k}{\rho}\right)^m \sum_{|q|=(k-1)m} \|D^q f\|_{\Omega_{\rho(1-1/k)}} \\
&= C_1 \sum_{|p|=m} \sum_{|q|=(k-2)m} \|D^p D^q A f\|_{\Omega_{\rho(1-1/k)}} \\
&\qquad + C_2\left(\frac{k}{\rho}\right)^m \sum_{|p|=m} \sum_{|q|=(k-2)m} \|D^p D^q f\|_{\Omega_{\rho(1-1/k)}} \\
&\leqq C_1^2 \sum_{|q|=(k-2)m} \|AD^q A f\|_{\Omega_{\rho(1-2/k)}} + \cdots \qquad \leqq \cdots \\
&\leqq C_1^k \|A^k f\|_\Omega + \binom{k}{1} C_1^{k-1} C_2 \left(\frac{k}{\rho}\right)^m \|A^{k-1} f\|_\Omega \\
&\qquad + \cdots + C_2^k \left(\frac{k}{\rho}\right)^{mk} \|f\|_\Omega \\
&\leqq \left(C_1 a^m + C_2 \frac{1}{\rho^m}\right)^k k^{mk} = (bk)^{mk}.
\end{aligned} \tag{12}
$$

Let j be an integer such that $mk \leqq j < m(k+1)$. Then

$$
\begin{aligned}
|||f|||_{j,\Omega_{2\rho}} &\leqq \sum_{|p|=j-mk} \sum_{|q|=mk} \|D^p D^q f\|_{\Omega_{2\rho}} \\
&\leqq C_3 \sum_{|q|=mk} \|AD^q f\|_{\Omega_\rho}^{(j-mk)/m} \|D^q f\|_{\Omega_\rho}^{1-(j-mk)/m} + C_4 \sum_{|q|=mk} \frac{1}{\rho^{j-mk}} \|D^q f\|_{\Omega_\rho} \\
&\leqq C_5(|||f|||_{m(k+1),\Omega_\rho} + |||f|||_{mk,\Omega_\rho}) \\
&\leqq C_6((b(k+1))^{m(k+1)} + (bk)^{mk}) \leqq M(cj)^j,
\end{aligned} \tag{13}
$$

where $M = C_6(b^m+1)$ and $c = 2b$, which are independent of j. Thus f is analytic in $\Omega_{2\rho}$ by Proposition 1.

3. Application. In this section we consider the solution of a parabolic differential equation

$$
\frac{\partial}{\partial t} u(t,x) = (-1)^{m+1} A_x u(t,x), \quad t \geqq 0, \quad x \in D \subset R, \tag{14}
$$

where A is a strongly elliptic differential operator in x of order $2m$.

Lax-Milgram [3] treated this equation as an equation of evolution in $L^2(D)$ in case D is a bounded domain, and, applying the Hille-Yosida theory, proved the existence and regularity of the solution for the initial-boundary value problem. (D may be unbounded. See, e.g. Yosida [8].) The solution is given by the formula

$$
u(t,x) = T_t u_0, \tag{15}
$$

where T_t is the semi-group of bounded operators with infinitesimal generator A, and $u_0(x)$ is the initial value in $L_x^2(D)$.

K. Yosida [7, 8] proved, moreover, that $T_t u_0$ $(t>0)$ is strongly differentiable in t any number of times (also analytic) as an L^2-valued function and its derivatives $T_t^{(k)}u_0=(-1)^{(m+1)k}A^kT_tu_0$ satisfy, for a constant c, the inequality that

(16) $$\|A^kT_tu_0\| \leqq (ck/t)^k\|u_0\|.$$

Therefore the solution T_tu_0 satisfies the condition (3). Thus by Nelson's result we have

Proposition 2. For every $u_0 \in L_x^2(D)$, if $t>0$, the solution T_tu_0 of (14) is an analytic function in x in the domain where the coefficients of A are all analytic.

$u(t,x)=T_tu_0$ may be regarded as a locally square integrable function in $(t>0)\times D$, and $\partial^{2m}/\partial t^{2m}+A$ is an elliptic differential operator. Remembering

(17) $$(\partial^{2m}/\partial t^{2m}+A)^kT_tu_0=(A^{2m}+A)^kT_tu_0,$$

we see that, for any $t_2>t_1>0$,

(18) $$\|(\partial^{2m}/\partial t^{2m}+A)^kT_tu_0\|_{(t_1,t_2)\times D} \leqq (t_2-t_1)(c'mk/t_1)^{mk},$$

with a constant c'. Hence we obtain the following

Proposition 3. The solution T_tu_0 of (14) is analytic with respect to t and x in the domain $(t>0)\times D_0$, where D_0 is the domain in which the coefficients of A are all constant.

References

[1] K. O. Friedrichs: On the differentiability of the solutions of linear elliptic differential equations, Comm. Pure Appl. Math., **6**, 299-326 (1953).

[2] L. Hörmander: On interior regularity of solutions of partial differential equations, Comm. Pure Appl. Math., **11**, 197-218 (1958).

[3] P. D. Lax and A. Milgram: Parabolic Equations, Contributions to the Theory of Partial Differential Equations, Ann. Math. Studies, no. 33, 167-190, Princeton (1954).

[4] C. B. Morrey and L. Nirenberg: Analyticity of solutions of elliptic equations, Comm. Pure Appl. Math., **10**, 271-290 (1957).

[5] E. Nelson: Analytic vectors, Ann. of Math., **70**, 572-615 (1959).

[6] S. Soboleff: Sur un théorème d'analyse fonctionelle (in Russian), Mat. Sb., n. s., **4** (46), 471-497 (1938).

[7] K. Yosida: On the differentiability of semi-groups of linear operators, Proc. Japan Acad., **34**, 337-340 (1958).

[8] K. Yosida: An abstract analyticity in time for solutions of a diffusion equation, Proc. Japan Acad., **35**, 109-113 (1959).

[9] S. D. Eidelman: On fundamental solutions of parabolic systems (in Russian), Mat. Sb., n. s., **38** (80), 51-92 (1956).