24. *Fractional Powers of Infinitesimal Generators and the Analyticity of the Semi-groups Generated by Them*

By Kôsaku YOSIDA[1)]
Department of Mathematics, University of Tokyo
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1. Consider a one-parameter semi-group of bounded linear operators $T_t (t\geqq 0)$ on a Banach space X into X:

(1) $$T_t T_s = T_{t+s},\ T_0 = I \text{ (the identity operator)},$$

(2) $$\text{strong-}\lim_{t\to t_0} T_t x = T_{t_0} x,\ x\in X,$$

(3) $$\sup_t \| T_t \| < \infty.$$

The infinitesimal generator A of the semi-group T_t is defined by

(4) $$Ax = \text{strong-}\lim_{h\downarrow 0} h^{-1}(T_h - I)x.$$

It is known that A is a closed linear operator whose domain $D(A)$ is strongly dense in X. A fractional power

(5) $$-(-A)^\alpha,\ (0<\alpha<1),$$

of A was defined by S. Bochner[2)] and R. S. Phillips[3)] as the infinitesimal generator of the semi-group

(6) $$\hat{T}_t x = \hat{T}_{t,\alpha} x = \int_0^\infty T_\lambda x\, d\gamma_{t,\alpha}(\lambda),$$

where the measure $d\gamma_{t,\alpha}(\lambda)\geqq 0$ is defined through the Laplace integral

(7) $$\exp(-ta^\alpha) = \int_0^\infty \exp(-\lambda a) d\gamma_{t,\alpha}(\lambda),\ (t, a>0 \text{ and } 0<\alpha<1).$$

The purpose of the present note is to prove that this semi-group $\hat{T}_t = \hat{T}_{t,\alpha}$ is analytic in t,[4)] or more precisely, that $\hat{T}_t$ belongs to the class of semi-groups introduced in a previous note.[5)]

For any $x\in X$ and for any $t>0$, $\hat{T}_t x = \hat{T}_{t,\alpha} x$ is strongly differentiable in t, and $\hat{T}'_t x = \text{strong-}\lim_{h\downarrow 0} h^{-1}(\hat{T}_{t+h} - \hat{T}_t)x$ satisfies

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1) Dedicated to Prof. Zyoiti Suetuna on his 60th Birthday.

2) Diffusion equations and stochastic processes, Proc. Nat. Acad. Sci., **35**, 369-370 (1949).

3) On the generation of semi-groups of linear operators, Pacific J. Math., **2**, 343-369 (1952).

4) Originally the author proved the analyticity for the case $0<\alpha\leqq 1/2$. It was communicated to Prof. Tosio Kato, and he has proved the analyticity for the case $0<\alpha<1$ by a more general approach. See the following paper by Prof. Kato. The author wishes to express his hearty thanks to Prof. Kato for the friendly discussion.

5) K. Yosida: On the differentiability of semi-groups of linear operators, Proc. Japan Acad., **34**, 337-340 (1958). Cf. E. Hille's class $H(\Phi_1, \Phi_2)$ of semi-groups in his book: Functional Analysis and Semi-groups, New York (1948).

(8) $$\overline{\lim_{t\downarrow 0}}\, t\, \| \hat{T}_t' \| < \infty,$$

so that[6] *the semi-group* $\hat{T}_t$ *can, as an abstract function of* t, *be extended analytically into a sector of the complex* λ*-plane defined by*

(9) $|\lambda - t| < Ct$, *where* C *is a positive constant.*

Remark 1. The proof given below in 2 is based upon an explicit representation of the semi-group $\hat{T}_t$: For any θ with $\pi/2 \leqq \theta \leqq \pi$, we have

(10) $$\hat{T}_t x = \hat{T}_{t,\alpha} x = \int_0^\infty f_{t,\alpha}(\lambda) T_\lambda x \, d\lambda, \; x \in X,$$

where

(11) $$f_{t,\alpha}(\lambda) = \pi^{-1} \int_0^\infty \exp(\lambda r \cdot \cos\theta - t r^\alpha \cos\alpha\theta)[\sin(\lambda r \cdot \sin\theta - t r^\alpha \sin\alpha\theta + \theta)] dr.$$

From this representation we easily derive the following formulae for $-(A)^\alpha$, announced recently by A. V. Balakrishnan:[7]

(12) $$-(-A)^\alpha x = (-\Gamma(-\alpha))^{-1} \int_0^\infty \lambda^{-\alpha-1}(T_\lambda - I) x \, d\lambda, \; x \in D(A),$$

(13) $$-(-A)^\alpha x = \pi^{-1} \sin\alpha\pi \int_0^\infty \lambda^{\alpha-1}(\lambda I - A)^{-1} A x \, d\lambda, \; x \in D(A).$$

Remark 2. Let A, $-A$ and A^2 be infinitesimal generators of semi-groups. Then a "Hilbert transform C_A associated with A" shall be defined through

(14) $$\begin{aligned} C_A \cdot Ax &= -(-A^2)^{1/2} x = \pi^{-1} \int_0^\infty \lambda^{-1/2} (\lambda I - A^2)^{-1} A^2 \cdot x \, d\lambda \\ &= \pi^{-1} \int_0^\infty 2^{-1} \lambda^{-1/2} \{(\lambda^{1/2} I - A)^{-1} - (\lambda^{1/2} I + A)^{-1}\} A \cdot x \, d\lambda \\ &= \pi^{-1} \int_0^\infty \{(\lambda I - A)^{-1} - (\lambda I + A)^{-1}\} A x \, d\lambda, \; x \in D(A^2). \end{aligned}$$

This definition is suggested by the following situation: Let $(Ax)(s) = dx(s)/ds$ for $x(s) \in C[-\infty, \infty]$. Then

$$((\lambda I - A)^{-1} x)(s) = \int_0^\infty \exp(-\lambda t) x(s+t) \, dt,$$

$$((\lambda I + A)^{-1} x)(s) = \int_0^\infty \exp(-\lambda t) x(s-t) \, dt,$$

so that

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6) See the note referred to in 5).

7) Representation of abstract Riesz potentials of the elliptic type, Bull. Amer. Math. Soc., **64**, no. 5, 288–289 (1958). Fractional powers of closed operators and the semi-groups generated by them, ibid., abstract, no. 558–23 (1959). Cf. M. A. Krasnoselski and P. E. Sobolevski: Fractional power operators defined on Banach spaces (in Russian), Doklady Academy Nauk, **129**, no. 3, 499–502 (1959).

$$
\begin{aligned}
[-(-A^2)^{1/2}x](s) &= \pi^{-1} \lim_{\varepsilon \downarrow 0} \int_0^\infty d\lambda \left\{ \int_\varepsilon^\infty \exp(-\lambda t)(x'(s+t)-x'(s-t))dt \right\} \\
&= \pi^{-1} \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty t^{-1}(x'(s+t)-x'(s-t))\, dt \qquad (15) \\
&= \text{the Hilbert transform of } (Ax)(s).
\end{aligned}
$$

[8]

2. We shall give the proof of the result in 1. Inverting the Laplace integral (7), we see that the measure $d\gamma_{t,\alpha}(\lambda)$ has the density $f_{t,\alpha}(\lambda)$ given by

$$
f_{t,\alpha}(\lambda) = (2\pi i)^{-1} \int_{\sigma - i\infty}^{\sigma + i\infty} \exp(z\lambda - z^\alpha t) dz \quad (\text{for any } \sigma > 0). \tag{16}
$$

Hence we obtain (10)–(11) by deforming the path of integration in (16) to the union of two paths: $r \cdot \exp(-i\theta)$ $(\infty > r > 0)$ and $r \cdot \exp(i\theta)$ $(0 < r < \infty)$. Taking $\theta = \theta_\alpha = \pi/(1+\alpha)$ in (10)–(11) and differentiating with respect to t, we obtain

$$
\hat{T}'_t x = \pi^{-1} \int_0^\infty T_\lambda x\, d\lambda \left\{ \int_0^\infty \exp((\lambda r + t r^\alpha)\cos\theta_\alpha) \cdot [\sin((\lambda r - t r^\alpha)\sin\theta_\alpha)] r^\alpha dr \right\}. \tag{17}
$$

This formal differentiation is justified, since the right hand side reduces, upon changing the variables of integration, to

$$
(t\pi)^{-1} \int_0^\infty T_{\nu t^{1/\alpha}} \cdot x\, d\nu \left\{ \int_0^\infty \exp((s\nu + s^\alpha)\cos\theta_\alpha)[\sin((s\nu - s^\alpha)\sin\theta_\alpha)]\, s^\alpha ds \right\}, \tag{18}
$$

which is, by $\cos\theta_\alpha < 0$ and (3), uniformly convergent in $t \geqq t_0$ for any fixed $t_0 > 0$. At the same time we have proved (8).

By a similar argument as above, we see, by (7), that

$$
\int_0^\infty (\partial f_{t,\alpha}(\lambda)/\partial t)\, d\lambda = 0. \tag{19}
$$

Hence we obtain, from (17),

$$
\begin{aligned}
\hat{T}'_t x = \pi^{-1} \int_0^\infty (T_\lambda - I)x\, d\lambda \Big\{ \int_0^\infty & \exp((\lambda r + t r^\alpha)\cos\theta_\alpha) \\
& \cdot [\sin((\lambda r - t r^\alpha)\sin\theta_\alpha)] r^\alpha\, dr \Big\}.
\end{aligned} \tag{20}
$$

If $x \in D(A)$, then $\lim_{\lambda \downarrow 0} \|(T_\lambda - I)\lambda^{-1} x\| = \|Ax\|$ and $\overline{\lim}_{\lambda \to \infty} \|(T_\lambda - I)x\| < \infty$. Thus we obtain, by letting $t \downarrow 0$ in (20),

$$
\begin{aligned}
\text{strong-}\lim_{t \downarrow 0} T'_t x &= \pi^{-1} \int_0^\infty (T_\lambda - I)x\, d\lambda \Big\{ \int_0^\infty \exp(\lambda r \cdot \cos\theta_\alpha) \\
& \qquad \cdot [\sin(\lambda r \cdot \sin\theta_\alpha)] r^\alpha dr \Big\} \qquad (21) \\
&= (-\Gamma(-\alpha))^{-1} \int_0^\infty \lambda^{-\alpha-1}(T_\lambda - I)x\, d\lambda,
\end{aligned}
$$

because

8) Cf. p. 605 in E. Hille and R. S. Phillips: Functional Analysis and Semi-groups, Providence (1957).

$$\begin{aligned}
&\pi^{-1}\int_0^\infty \exp(\lambda r\cdot\cos\theta_\alpha)\cdot\sin(\lambda r\cdot\sin\theta_\alpha)r^\alpha dr\\
(22)\qquad &=(2\pi)^{-1}\cdot i\cdot\Gamma(1+\alpha)[(-\lambda\cos\theta_\alpha+i\lambda\sin\theta_\alpha)^{-1-\alpha}\\
&\quad-(-\lambda\cos\theta_\alpha-i\lambda\sin\theta_\alpha)^{-1-\alpha}]\\
&=(-\Gamma(-\alpha))^{-1}\lambda^{-\alpha-1}.
\end{aligned}$$

Therefore (12) is proved by $\hat{T}'_t x=(-(-A)^\alpha)\hat{T}_t x$ $(t>0)$, the strong continuity in t of $\hat{T}_t$ and the closure property of the infinitesimal generator $-(-A)^\alpha$.

Lastly, by making use of

$$\Gamma(1+\alpha)\lambda^{-\alpha-1}=\int_0^\infty \exp(-\lambda t)t^\alpha\, dt$$

and the resolvent formula

$$(23)\qquad (\lambda I-A)^{-1}x=\int_0^\infty \exp(-\lambda t)T_t x dt,\ x\in X,$$

in semi-group theory, we obtain (13) from (12) because of

$$(\lambda I-A)^{-1}\cdot A\cdot x=\{\lambda(\lambda I-A)^{-1}-I\}x,\ \ x\in D(A).$$

Remark 3. If we take $\theta=\pi$ in (10)–(11) and make use of (23) to the semi-group $\hat{T}_t$, we obtain another proof of the formula due to T. Kato:[9]

$$\begin{aligned}
&(\mu+(-A)^\alpha)^{-1}=\int_0^\infty \exp(-\mu t)\,\hat{T}'_{t,\alpha}dt\\
(24)\quad &=\pi^{-1}\int_0^\infty dr\int_0^\infty \exp(-\lambda r)T_\lambda d\lambda\int_0^\infty \exp(-\mu t-tr^\alpha\cos\alpha\pi)\sin(tr^\alpha\sin\alpha\pi)dt\\
&=\frac{\sin\alpha\pi}{\pi}\int_0^\infty (r-A)^{-1}\frac{r^\alpha}{\mu^2-2\mu r^\alpha\cos\alpha\pi+r^{2\alpha}}dr.
\end{aligned}$$

9) See Kato's paper referred to in 4).