23. *Cosheaves*

By Yukiyosi KAWADA
Department of Mathematics, University of Tokyo
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In this note we shall define a cosheaf on a paracompact space X, which is a dual concept of a sheaf (§1). If the base space X is a compact Hausdorff space we can develop a homology theory of X with coefficients in a cosheaf (§2). This homology theory is equivalent to the Čech homology theory and is dual to the cohomology theory with coefficients in a sheaf (§3).[1]

1. Let X be a paracompact space. We denote by $\mathfrak{A}(X)$ the family of all closed subsets of X. Let us suppose that a compact topological additive group $\mathfrak{F}(A)$ is associated with each $A\in\mathfrak{A}(X)$. In particular, let $\mathfrak{F}(\phi)=\{0\}$. For each pair (A, B) $(A, B\in\mathfrak{A}(X)$ and $A\supset B)$ let a continuous homomorphism $\iota_{A,B}$ of $\mathfrak{F}(B)$ into $\mathfrak{F}(A)$ be defined such that (i) $\iota_{A,A}$ is the identity mapping for each $A\in\mathfrak{A}(X)$, and (ii) $\iota_{A,C}=\iota_{A,B}\circ\iota_{B,C}$ holds for $A, B, C\in\mathfrak{A}(X)$ and $A\supset B\supset C$. Moreover, let $\mathfrak{U}(A)$ $(A\in\mathfrak{A}(X))$ be the family of all $B\in\mathfrak{A}(X)$ such that A is contained in the interior of B. Then $\mathfrak{U}(A)$ is a directed family of sets with respect to the inclusion relation and $\{\mathfrak{F}(B); B\in\mathfrak{U}(A)\}$ is an inverse system of compact additive groups with respect to the continuous homomorphisms $\{\iota\}$. Let us suppose further that

$$(1)\qquad \mathfrak{F}(A)=\text{inv lim}\{\mathfrak{F}(B); B\in\mathfrak{U}(A)\}\qquad \text{for } A\in\mathfrak{A}(X)$$

hold. Then we call the system $\mathfrak{F}=\{\mathfrak{F}(A), \iota_{A,B}\}$ a *precosheaf* with the base space X.[2] If necessary we denote $\iota^{\mathfrak{F}}_{A,B}$ instead of $\iota_{A,B}$. In the following we fix a base space X.

A precosheaf $\mathfrak{G}=\{\mathfrak{G}(A), \iota^{\mathfrak{G}}_{A,B}\}$ is called a *subprecosheaf* if (i) for each $A\in\mathfrak{A}(X)$ $\mathfrak{G}(A)$ is a closed subgroup of $\mathfrak{F}(A)$ with the relative topology, (ii) $\iota^{\mathfrak{G}}_{A,B}=\iota^{\mathfrak{F}}_{A,B}|\mathfrak{G}(B)$ for $A\supset B$ holds and (iii) $\mathfrak{G}(A)=\text{inv lim}\{\mathfrak{G}(B); B\in\mathfrak{U}(A)\}$ holds for each $A\in\mathfrak{A}(X)$. Let $\mathfrak{G}$ be a subprecosheaf of a precosheaf $\mathfrak{F}$. Let us put $\mathfrak{H}(A)=\mathfrak{F}(A)/\mathfrak{G}(A)$ with the quotient topology for each $A\in\mathfrak{A}(X)$ and let the homomorphism $\iota^{\mathfrak{H}}_{A,B}$ be induced from $\iota^{\mathfrak{F}}_{A,B}$. Then $\mathfrak{H}=\{\mathfrak{H}(A), \iota^{\mathfrak{H}}_{A,B}\}$ is a precosheaf. We call $\mathfrak{H}$ the *quotient precosheaf* of $\mathfrak{F}$ by $\mathfrak{G}$.

Let $\mathfrak{F}, \mathfrak{G}$ be two precosheaves. Let φ_A be a continuous homomorphism of $\mathfrak{F}(A)$ into $\mathfrak{G}(A)$ for each $A\in\mathfrak{A}(X)$ and let us assume that

1) In this note we shall only sketch our results. The details and further developments will be discussed in another paper.

2) This definition is dual to that of a sheaf used in Cartan [1], XII: Faisceaux et carapaces.

$\iota^{\mathfrak{G}}_{A,B}\circ\varphi_B=\varphi_A\circ\iota^{\mathfrak{F}}_{A,B}$ holds for each pair (A, B) $(A, B\in\mathfrak{A}(X)$ and $A\supset B)$. Then we call the system $\varphi=\{\varphi_A; A\in\mathfrak{A}(X)\}$ a *homomorphism* of $\mathfrak{F}$ into $\mathfrak{G}$ and we denote $\varphi:\mathfrak{F}\to\mathfrak{G}$. The image of φ and the kernel of φ can be defined naturally which are subprecosheaves of $\mathfrak{G}$ and $\mathfrak{F}$ respectively.

A precosheaf $\mathfrak{F}$ is called *locally zero* if the following condition is satisfied: Let $s\in\mathfrak{F}(A)$, $A\in\mathfrak{A}(X)$. If $s\in\sum_{i=1}^n\iota_{A,A_i}\mathfrak{F}(A_i)$ holds for any finite closed covering $\{A_1,\cdots,A_n\}$ of A, then $s=0$.

A precosheaf $\mathfrak{F}$ is called a *cosheaf* if the following two conditions (F1) and (F2) are satisfied.

(F1) Let $\{A_1,\cdots,A_n\}$ be a finite closed covering of $A\in\mathfrak{A}(X)$. Then

$$\mathfrak{F}(A)=\sum_{i=1}^n\iota_{A,A_i}\mathfrak{F}(A_i)$$

holds.

(F2) Let $\{A_1,\cdots,A_n\}$ be a finite closed covering of $A\in\mathfrak{A}(X)$. If $s_i\in\mathfrak{F}(A_i)$ $(i=1,\cdots,n)$ satisfy the equality $\sum_{i=1}^n\iota_{A,A_i}s_i=0$, then there exist $s_{ij}\in F(A_i\cap A_j)$ $(i,j=1,\cdots,n)$ such that (i) $s_{ii}=0$, (ii) $s_{ij}=-s_{ji}$ and (iii) $s_i=\sum_{j=1}^n\iota_{A_i,A_i\cap A_j}s_{ji}$ hold for $i,j=1,\cdots,n$.

PROPOSITION 1. *Let $\mathfrak{F}$ be an arbitrary precosheaf. Then there exist precosheaves $\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$ such that* (i) $0\to\mathfrak{F}_1\to\mathfrak{F}\to\mathfrak{F}_0\to0$ *and* $0\to\mathfrak{F}_3\to\mathfrak{F}_2\to\mathfrak{F}_1\to0$ *are exact sequences,* (ii) $\mathfrak{F}_0, \mathfrak{F}_3$ *are locally zero precosheaves,* (iii) $\mathfrak{F}_1$ *satisfies the condition* (F1) *and* (iv) $\mathfrak{F}_2$ *is a cosheaf. Moreover, these precosheaves $\mathfrak{F}_i$ $(i=0,1,2,3)$ are uniquely determined by $\mathfrak{F}$ up to isomorphism.*

We denote then $\mathfrak{F}_2=\Gamma\mathfrak{F}$ and $\Gamma\mathfrak{F}$ is called the *cosheaf generated* by $\mathfrak{F}$.

Let $\mathfrak{F}, \mathfrak{G}$ be cosheaves and let $\varphi:\mathfrak{F}\to\mathfrak{G}$ be a homomorphism such that image $\varphi=\mathfrak{G}$ holds. Then kernel $\varphi=\mathfrak{H}'$ is a precosheaf satisfying the condition (F1) but not necessarily (F2). Let $\mathfrak{H}=\Gamma\mathfrak{H}'$ be the cosheaf generated by $\mathfrak{H}'$. We denote then

(2) $$0\longrightarrow\mathfrak{H}\longrightarrow\!\!\!\!\!\rightarrow\mathfrak{F}\longrightarrow\mathfrak{G}\longrightarrow0 \qquad \text{(exact)}.$$

Namely, (2) is equivalent to an exact sequence of precosheaves:

(3) $$0\longrightarrow\mathfrak{H}_0\longrightarrow\mathfrak{H}\longrightarrow\mathfrak{F}\longrightarrow\mathfrak{G}\longrightarrow0$$

where $\mathfrak{H}_0$ is a locally zero precosheaf.

In general, let $\{\mathfrak{F}_n\}$ be a sequence of cosheaves and let $\{\varphi_n:\mathfrak{F}_n\to\mathfrak{F}_{n-1}\}$ be a sequence of homomorphisms. We denote then

$$\cdots\longrightarrow\!\!\!\!\!\rightarrow\mathfrak{F}_{n+1}\xrightarrow{\varphi_{n+1}}\mathfrak{F}_n\xrightarrow{\varphi_n}\mathfrak{F}_{n-1}\longrightarrow\!\!\!\!\!\rightarrow\cdots \qquad \text{(exact)}$$

if (i) image $\varphi_n=\mathfrak{G}_n$ are all cosheaves and (ii) $0\to\mathfrak{G}_{n+1}\longrightarrow\!\!\!\!\!\rightarrow\mathfrak{F}_n\to\mathfrak{G}_n\to0$ are exact sequences in the sense of (2) for all n.

The *stalk* of a precosheaf $\mathfrak{F}$ at x $(x\in X)$ is simply $\mathfrak{F}(\{x\})$ which we denote by $\mathfrak{F}(x)$. Then we see easily the following properties on stalks. (i) Let $\mathfrak{F}$ be an arbitrary precosheaf. Then for every $x\in X$ $\mathfrak{F}(x)\cong\Gamma\mathfrak{F}(x)$ holds. (ii) If $\mathfrak{F}$ is locally zero then $\mathfrak{F}(x)=\{0\}$ for every $x\in X$. (iii) Let $\mathfrak{F}, \mathfrak{G}$ and $\mathfrak{H}$ be cosheaves such that $0\to\mathfrak{F}\longrightarrow\!\!\!\!\!\rightarrow\mathfrak{G}\to\mathfrak{H}\to0$ (exact).

Then for every $x\in X$ we have an induced exact sequence $0\to\mathfrak{F}(x)\to\mathfrak{G}(x)\to\mathfrak{H}(x)\to 0$.

2. In the following we assume always that *the base space X is a compact Hausdorff space.* A precosheaf $\mathfrak{F}$ is called *complete*[3] if every $\iota_{A,B}(A, B\in\mathfrak{A}(X)$ and $A\supset B)$ is a monomorphism. If we identify $\mathfrak{F}(A)$ with the subgroup $\iota_{X,A}\mathfrak{F}(A)$ of $\mathfrak{F}(X)$ then a complete precosheaf $\mathfrak{F}=\{\mathfrak{F}(A); A\in\mathfrak{A}(X)\}$ consists of a family of closed subgroups of $\mathfrak{F}(X)$ such that $A\supset B$ implies $\mathfrak{F}(A)\supset\mathfrak{F}(B)$ and (1) holds.

Now we shall define a complete cosheaf $\tilde{\mathfrak{F}}$ which is canonically associated with a given cosheaf $\mathfrak{F}$. Let $\mathfrak{F}^*(X)$ be the direct sum of (abstract) additive groups $\mathfrak{F}(x)$ for all $x\in X$. Then we shall define a precompact topology in $\mathfrak{F}^*(X)$ as follows. Let $s=\sum_i s_{x_i}$, $s_{x_i}\in\mathfrak{F}(x_i)$, $x_i\in X$ be an element of $\mathfrak{F}^*(X)$. Let us take arbitrarily a finite number of continuous characters $\chi_x^{(k)}$ $(k=1,\cdots,n)$ of the compact additive group $\mathfrak{F}(x)$ for each $x\in X$ and let us put $\chi^{(k)}=\{\chi_x^{(k)}; x\in X\}$. Let $\varepsilon>0$. Then the neighbourhood $U(s;\chi^{(1)},\cdots,\chi^{(n)},\varepsilon)$ of $s\in\mathfrak{F}^*(X)$ be defined as the totality of all $t=\sum_j t_{y_j}\in\mathfrak{F}^*(X)$ $(t_{y_j}\in\mathfrak{F}(y_j), y_j\in X)$ such that

$$\Big|\sum_i \chi_{x_i}^{(k)}(s_{x_i})-\sum_j \chi_{y_j}^{(k)}(t_{y_j})\Big|<\varepsilon \qquad (k=1,\cdots,n)$$

hold. If we take all these $U(s;\chi^{(1)},\cdots,\chi^{(n)},\varepsilon)$ as the fundamental system of neighbourhoods of s in $\mathfrak{F}^*(X)$ we see that $\mathfrak{F}^*(X)$ is a precompact topological group. Then we define the compact additive group $\tilde{\mathfrak{F}}(X)$ as the completion of $\mathfrak{F}^*(X)$. For $A\in\mathfrak{A}(X)$ we define the closed subgroup $\tilde{\mathfrak{F}}(A)$ of $\tilde{\mathfrak{F}}(X)$ as follows. Let $\{\chi_x; x\in X\}$ be a system of continuous characters of $\mathfrak{F}(x)$ $(x\in X)$ such that $\chi_x=0$ hold for all $x\in U$ where U is some open set containing A. We denote by $\mathfrak{J}(A)$ the set of all such systems $\{\chi_x\}$. Then $\tilde{\mathfrak{F}}(A)$ is the closed subgroup of $\tilde{\mathfrak{F}}(X)$ consisting of all elements which annihilate $\mathfrak{J}(A)$. We can verify that $\tilde{\mathfrak{F}}=\{\tilde{\mathfrak{F}}(A); A\in\mathfrak{A}(X)\}$ is a precosheaf.

PROPOSITION 2. *The above defined precosheaf $\tilde{\mathfrak{F}}$ is a complete cosheaf, and there exists a canonical homomorphism $\varphi:\tilde{\mathfrak{F}}\to\mathfrak{F}$ such that* image $\varphi=\mathfrak{F}$ *holds.*

By a *complete resolution* of a cosheaf $\mathfrak{F}$ we mean an exact sequence of cosheaves $\mathfrak{F}_n$ and homomorphisms $\varphi_n:\mathfrak{F}_n\to\mathfrak{F}_{n-1}$:

(4) $\qquad \cdots\longrightarrow\mathfrak{F}_n\longrightarrow\mathfrak{F}_{n-1}\longrightarrow\cdots\longrightarrow\mathfrak{F}_1\longrightarrow\mathfrak{F}_0\longrightarrow\mathfrak{F}\longrightarrow 0$ (exact)

where $\mathfrak{F}_n$ $(n=0,1,2,\cdots)$ are all complete. The existence of such a complete resolution of a given cosheaf $\mathfrak{F}$ follows from Proposition 2. Let (4) be a complete resolution of a given cosheaf $\mathfrak{F}$. Then the sequence of compact additive groups $\mathfrak{F}_n(X)$ and continuous homomorphisms $\partial_n:\mathfrak{F}_n(X)\to\mathfrak{F}_{n-1}(X)$ $(\partial_n=(\varphi_n)_X)$:

3) The concept of a complete precosheaf is the dual of "faisceau mou" of Godement [2]. We use the terminology "complete" after Sato [4].

$$\cdots \longrightarrow \mathfrak{F}_n(X) \xrightarrow{\partial_n} \mathfrak{F}_{n-1}(X) \longrightarrow \cdots \longrightarrow \mathfrak{F}_1(X) \xrightarrow{\partial_1} \mathfrak{F}_0(X) \xrightarrow{\partial_0} 0$$

satisfies $\partial_{n-1} \circ \partial_n = 0$ $(n=1, 2, \cdots)$. Hence we can define the *homology groups of X with coefficients in a cosheaf $\mathfrak{F}$* by

$$H_n(X, \mathfrak{F}) = \text{kernel } \partial_n / \text{image } \partial_{n+1} \qquad (n=0, 1, 2, \cdots)$$

which are compact groups.

PROPOSITION 3. (i) *The above defined homology groups $H_n(X, \mathfrak{F})$ for a cosheaf $\mathfrak{F}$ are independent of the choice of a complete resolution* (4) *of $\mathfrak{F}$.*

(ii) $H_0(X, \mathfrak{F}) \cong \mathfrak{F}(X)$.

(iii) *If $\mathfrak{F}$ is a complete cosheaf then $H_n(X, \mathfrak{F}) = 0$ for $n=1, 2, \cdots$.*

(iv) *Each homomorphism $\varphi : \mathfrak{F} \to \mathfrak{G}$ of cosheaves induces continuous homomorphisms*

$$\varphi_n^* : H_n(X, \mathfrak{F}) \to H_n(X, \mathfrak{G}) \qquad (n=0, 1, 2, \cdots)$$

such that (α) *φ_n^* is the identity of $H_n(X, \mathfrak{F})$ if φ is the identity of $\mathfrak{F}$ and* (β) *for two homomorphisms $\varphi : \mathfrak{F} \to \mathfrak{G}$ and $\psi : \mathfrak{G} \to \mathfrak{H}$ $(\psi \circ \varphi)_n^* = \psi_n^* \circ \varphi_n^*$ holds.*

(v) *Let $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be cosheaves such that $0 \to \mathfrak{F} \longrightarrow\!\!\!\!\!\to \mathfrak{G} \to \mathfrak{H} \to 0$ (exact). Then there exist continuous homomorphisms*

$$\partial_n^* : H_n(X, \mathfrak{H}) \longrightarrow H_{n-1}(X, \mathfrak{F}) \qquad (n=1, 2, \cdots)$$

such that

$$\cdots \longrightarrow H_n(X, \mathfrak{F}) \longrightarrow H_n(X, \mathfrak{G}) \longrightarrow H_n(X, \mathfrak{H}) \xrightarrow{\partial_n^*} H_{n-1}(X, \mathfrak{F}) \longrightarrow \cdots$$
$$\longrightarrow H_1(X, \mathfrak{H}) \xrightarrow{\partial_1^*} H_0(X, \mathfrak{F}) \longrightarrow H_0(X, \mathfrak{G}) \longrightarrow H_0(X, \mathfrak{H}) \longrightarrow 0$$

is an exact sequence.

(vi) *If*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{F}_1 & \longrightarrow\!\!\!\!\!\to & \mathfrak{G}_1 & \longrightarrow & \mathfrak{H}_1 & \longrightarrow & 0 \qquad (\textit{exact}) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{F}_2 & \longrightarrow\!\!\!\!\!\to & \mathfrak{G}_2 & \longrightarrow & \mathfrak{H}_2 & \longrightarrow & 0 \qquad (\textit{exact}) \end{array}$$

is a commutative diagram of cosheaves then

$$\begin{array}{ccc} H_n(X, \mathfrak{H}_1) & \xrightarrow{\partial_n^*} & H_{n-1}(X, \mathfrak{F}_1) \\ \downarrow & & \downarrow \\ H_n(X, \mathfrak{H}_2) & \xrightarrow{\partial_n^*} & H_{n-1}(X, \mathfrak{F}_2) \end{array}$$

is also a commutative diagram.

(vii) *The compact homology groups $H_n(X, \mathfrak{F})$ $(n=0, 1, 2, \cdots)$ associated with each cosheaf $\mathfrak{F}$ are uniquely determined by the properties* (ii)–(vi).

3. Here we assume also that the base space X is a compact Hausdorff space. For any precosheaf $\mathfrak{F}$ we can naturally define the compact Čech homology groups $\check{H}_n(X, \mathfrak{F})$ of the space X with coefficients in $\mathfrak{F}$ just as in the case of compact constant coefficient groups with necessary modifications (cf. e.g. Eilenberg and Steenrod [3]).

PROPOSITION 4. *If* $\mathfrak{F}$ *is a locally zero precosheaf then* $\check{H}_n(X, \mathfrak{F})=0$ *for* $n=0, 1, 2, \cdots$.

PROPOSITION 5. *The Čech homology groups* $\check{H}_n(X, \mathfrak{F})$ $(n=0, 1, 2, \cdots)$ *with coefficients in a cosheaf* $\mathfrak{F}$ *satisfy all the properties* (ii)-(vi) *of Proposition* 3. *Hence we have the isomorphisms*

$$\check{H}_n(X, \mathfrak{F}) \cong H_n(X, \mathfrak{F}) \qquad (n=0, 1, 2, \cdots).$$

Now let $\mathfrak{S}$ be a sheaf of additive groups on X and let $\mathfrak{S}(U)$ be the discrete additive group associated with each open subset U of X (cf. e.g. Godement [2]). For any closed set A the discrete additive group $\mathfrak{S}(A)=\Gamma(A, \mathfrak{S})$ of sections of $\mathfrak{S}$ over A is isomorphic to the direct limit group of the system $\{\mathfrak{S}(U); U\supset A\}$. Let us consider the system of compact character groups

$$\mathfrak{F}(A)=\text{character group of } \mathfrak{S}(A) \qquad (A\in\mathfrak{A}(X))$$

with naturally defined continuous homomorphisms $\iota_{A,B}$ for $A\supset B$. Then it is easy to see that $\mathfrak{F}=\{\mathfrak{F}(A); \iota_{A,B}\}$ is a cosheaf. We shall call $\mathfrak{F}$ the dual cosheaf of the sheaf $\mathfrak{S}$.

PROPOSITION 6. *Let* $\mathfrak{S}$ *be a sheaf of discrete additive groups on* X *and let* $\mathfrak{F}$ *be the dual cosheaf of* $\mathfrak{S}$. *Then each of the discrete cohomology group* $H^n(X, \mathfrak{S})$ *with coefficients in a sheaf* $\mathfrak{S}$ *and the compact homology group* $H_n(X, \mathfrak{F})$ *with coefficients in the dual cosheaf* $\mathfrak{F}$ *is the character group of the other* $(n=0, 1, 2, \cdots)$.

REMARK. If we assume that each $\mathfrak{F}(A)$ $(A\in\mathfrak{A}(X))$ is a linearly compact vector space over a field K and each $\iota_{A,B}$ is a K-linear mapping then we have the similar results as above.

Added in proof. M. Sato has also developed the theory of cosheaves from a different point of view. Also the author has found a paper on the theory of precosheaves in the recent number of Arch. d. Math., R. Kultze: Dualität von Homologie- und Cohomologiegruppen in der Garbentheorie, Arch. d. Math., **10**, 438-442 (1959). This paper refers to a paper of E. Luft: Eine Verallgemeinerung der Čechschen Homologietheorie, Bonner math. Schriften, nr. 8 (1959).

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