50. On Characterizations of Projection Operators

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Let R be a lattice ordered linear space. A linear manifold M of R is said to be *normal*, if for any $a \in R$ we can find $x, y \in R$ such that

$$a = x + y \quad x \in M, \quad y \in M^{\perp} = \{y;\ x \perp y \text{ for } x \in M\}.$$

Such x depends only on a. So putting $Ta = x$ we can define an operator T from R to M. This operator is called a *projection operator* (cf. H. Nakano: Modulared Semi-ordered Linear Space, Tokyo (1950)).

Here, we will consider some characterizations of projection operators.

Theorem 1. *A linear operator T on R is a projection operator, if and only if it satisfies* (1), (2).

(1) $$T(Tx) = Tx$$

(2) $$0 \leqq Tx \leqq x \qquad \textit{for all } x \geqq 0.$$

Proof. Every projection operator is always linear and satisfies (1), (2) (cf. H. Nakano: Modulared Semi-ordered Linear Space, Tokyo (1950)).

Now, we suppose that a linear operator T satisfies conditions (1), (2). Putting $T^{\perp} = I - T$, $T^{\perp}$ is obviously linear and satisfies conditions (1), (2) too. When we consider two subsets of R

$$A = \{x;\ Tx = 0\} \quad \text{and} \quad B = \{x;\ T^{\perp}x = 0\},$$

we have $A = T^{\perp}R$, $B = TR$, because

$$T(T^{\perp}a) = T(a - Ta) = Ta - T(Ta) = Ta - Ta = 0,$$

for any $a \in R$, and hence $T^{\perp}a \in A$. On the other hand, we see

$$a = a - Ta = T^{\perp}a,$$

for every $a \in A$, therefore $A = T^{\perp}R$. We obtain $B = TR$ likewise.

Every linear operator T, subject to the condition (2), satisfies

$$T(x \overset{\smile}{\frown} y) = Tx \overset{\smile}{\frown} Ty.$$

Because we see first obviously

$$Tx \frown Ty \geqq T(x \frown y).$$

On the other hand, we have

$$x = Tx + T^{\perp}x \geqq Tx \frown Ty + T^{\perp}(x \frown y),$$
$$y = Ty + T^{\perp}y \geqq Tx \frown Ty + T^{\perp}(x \frown y)$$

and hence

$$x \frown y \geqq Tx \frown Ty + T^{\perp}(x \frown y),$$

that is,

$$T(x \frown y) \geqq Tx \frown Ty.$$

Therefore

$$T(x \frown y) = Tx \frown Ty.$$

Then we find easily

$$T(x \smile y) = Tx \smile Ty,$$

because T is linear.

For any $x \in A$, $y \in B$ we have

$$0 \leqq T(|x| \frown |y|) \leqq T|x| = Tx \smile (-Tx) = 0$$

and

$$0 \leqq T^{\perp}(|x| \frown |y|) \leqq T^{\perp}|y| = T^{\perp}y \smile (-T^{\perp}y) = 0,$$

therefore

$$|x| \frown |y| = 0.$$

Consequently $A \perp B$, that is, $TR \perp T^{\perp}R$. Thus T is a projection operator by definition.

Theorem 2. *A linear operator T on R is a projection operator if and only if it satisfies the following conditions.*

(1) $$T(Tx) = Tx$$

(2) $$Tx \perp (x - Tx) \qquad \textit{for all } x \in R.$$

Proof. A projection operator is linear and satisfies (1) (Theorem 1) and obviously it satisfies (2) by definitions. When a linear operator satisfies (1), (2), we can see easily

$$(|Tx| - |x|) \frown |Tx| \leqq |x - Tx| \frown |Tx| = 0$$

by the condition (2).
Therefore

$$|Tx| \leqq |x| \smile 0 = |x|,$$

and hence $Tx \leqq x$ for $x \geqq 0$. Likewise we can prove $T^{\perp}x \leqq x$ for $x \geqq 0$. Consequently T satisfies the condition of Theorem 1, and hence T is a projection operator.

Now we want to remark that the condition (2) in Theorem 1, may be replaced by one of the following conditions:

2′) $$y \frown Tx = Tx \frown Ty \qquad \text{for } x, y \geqq 0$$

2″) $$T|x| = |Tx| \frown |x|$$

2‴) $$y \frown Tx = x \frown Ty \qquad \text{for } x, y \geqq 0.$$

Because, putting $x=0$ or $x=y$ in 2′), we obtain $T \geqq 0$ or $T \leqq 1$ respectively. 2″) implies obviously $0 \leqq T \leqq 1$. Putting $y=0$ or $y=Tx$ in 2‴), we obtain $T \geqq 0$ or $T \leqq 1$ respectively.