49. *On Locally Compact Halfrings*

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1. Introduction. *Definition* 1. A halfring H is a semiring which can be embedded in a ring.

Since we shall confine ourselves to a semiring with commutative addition, it is necessary and sufficient that the cancellation law of addition holds in this semiring for it to be a halfring.

The product set $H\times H$ forms again a halfring according to the laws

$$(1)\qquad \begin{aligned}(i_1, j_1)+(i_2, j_2)&=(i_1+i_2, j_1+j_2),\\ (i_1, j_1)(i_2, j_2)&=(i_1i_2+j_1j_2, i_1j_2+j_1i_2).\end{aligned}$$

The diagonal $\varDelta=\{(x, x)|x\in H\}$ of H is a two-sided ideal in $H\times H$.

Definition 2. Two elements (i_1, j_1), (i_2, j_2) of the halfring $H\times H$ are said to be equivalent modulo $\varDelta$, if there exist elements (x, x) and (y, y) in $\varDelta$ such that

$$(2)\qquad (i_1, j_1)+(x, x)=(i_2, j_2)+(y, y).$$

This equivalence relation is a special case of the one given in a previous paper [1]. From (2) we obtain that $i_1+x=i_2+y$, $j_1+x=j_2+y$, $i_1+x+j_2+y=i_2+y+j_1+x$ and $i_1+j_2=i_2+j_1$. Also, if $i_1+j_2=i_2+j_1$ then $(i_1, j_1)+(j_2, j_2)=(i_2, j_2)+(j_1, j_1)$ and $(i_1, j_1)\sim(i_2, j_2)$. The difference ring $R=H\times H/\varDelta$ is defined to be the ring generated by H. Let ν denote the natural homomorphism of $H\times H$ onto R, then the halfring H is embedded in the ring R, for the mapping $h\to\nu(h+a, a)$, for any a, is an isomorphism of H into R.

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2. Quotient spaces. *Definition* 3. A topological halfring is a halfring H together with a Hausdorff topology on H under which the halfring operations are continuous.

We introduce in R the quotient topology, that is the largest topology for R such that the projection (quotient map) ν is a continuous mapping of $H\times H$ onto R. We assume that H is a locally compact space, then $H\times H$ is locally compact in the product topology. If the projection ν is open, that is the image of each open set is open, then R is also a locally compact space [5]. Hence, it will be fruitful to impose a property on H which will insure that ν be an open mapping. Furthermore, if ν is open then R is a topological ring [3]. In a recent paper [6], N. J. Rothman imposes such a topological and algebraic *property F* on a commutative semigroup with cancellation. We shall likewise adopt this property.

Definition 4. A halfring has property F if i, $x \in H$ and an open subset $V_i \ni i$ imply that there exists an open subset $U_x \ni x$ such that $x+i \in \bigcap [V_i + x' | x' \in U_x]$.

In any case, this condition implies that translations are open mappings. However, if H possesses a zero this is equivalent to saying that there exists a neighborhood of zero such that $U = -U$ and negation is continuous in U [6]. The topological halfring of positive reals $R^+ + 1$ with the usual topology satisfies condition F, while $R^+ \cup \{0\}$ does not.

LEMMA 1. *If H has property F then $H \times H$ has property F.*

Proof. Since H has property F, there exist open subsets U_x and U_y in H such that $x+i \in \bigcap [V_i + x' | x' \in U_x]$ and $y+j \in \bigcap [V_j + y' | y' \in U_y]$. Hence, for $(i, j), (x, y) \in H \times H$ and an open subset $V_{ij} = V_i \times V_j$, there exists an open subset $U_{xy} = U_x \times U_y$ such that $(x, y) + (i, j) \in \bigcap [V_{ij} + (x', y') | (x', y') \in U_{xy}]$.

For the sake of completeness, we repeat for a topological halfring the proof given by N. J. Rothman for a commutative topological semigroup with cancellation [6].

LEMMA 2. *If H has property F, then the projection ν is an open mapping.*

Proof. Let A be an open subset of $H \times H$ and $\nu(A)$ its image. We show that $\nu(A)$ is an open subset of R. Since R is a quotient space of $H \times H$, this is equivalent to proving that the set $\widehat{A} = \nu^{-1}\nu(A)$ is an open subset of $H \times H$. Let (x, y) be in $\widehat{A}$. We wish to show that there exists an open subset U_{xy} of (x, y) in $H \times H$, such that $U_{xy} \subset \widehat{A}$. There exists an $(i, j) \in A$ such that $\nu(x, y) = \nu(i, j)$, that is $(x, y) + (j, j) = (i, j) + (y, y)$. Since A is an open subset of $H \times H$, there exists an open subset V_{ij} of (i, j) in $H \times H$ such that $V_{ij} \subset A$. By Lemma 1 there exists an open subset U_{xy} such that $(x, y) + (j, i) \in (x', y') + V_{ji}$ for all $(x', y') \in U_{xy}$. For $(x', y') \in U_{xy}$, we have a $(j', i') \in V_{ji}$ such that $(x, y) + (j, i) = (x', y') + (j', i')$. Because $(x+j, y+j) = (i+y, j+y)$, we have that $(x'+j', y'+j') = (i'+y', j'+y')$ or $(x', y') + (j', j') = (i', j') + (y', y')$. Thus $\nu(x', y') = \nu(i', j')$ and $(x', y') \in \widehat{A}$, for all $(x', y') \in U_{xy}$. This implies that $U_{xy} \subset \widehat{A}$, $\widehat{A}$ an open subset of $H \times H$ and the mapping ν is open.

As a consequence of Lemma 2, we have

THEOREM 1. *A locally compact topological halfring with property F is embeddable in a locally compact topological ring.*

3. Bounded halfrings. We further impose that H possesses a zero element, although this may not be necessary. In agreement with

Shafarevich [7] we give

Definition 5. A subset S of a topological halfring H is said to be right bounded if for any open neighborhood U of 0 there exists an open neighborhood V of 0 such that $V\cdot S=\{vs|v\in V, s\in S\}$ is contained in U.

Left boundedness is similarly defined. When H is both right bounded and left bounded, it is said to be bounded.

LEMMA 3. *If H is right bounded then $R=H\times H/\Delta$ is right bounded.*

Proof. Let $V\in R$ be an open neighborhood of the zero of R. It follows that $v^{-1}(V)$ is an open neighborhood of Δ in $H\times H$. Consequently there exist open sets U_1, U_2 of 0 in H such that $(0,0)\in U_1\times U_2\subset\nu^{-1}(V)$. We choose W_1 and W_2 to be open neighborhoods of 0 so that $W_1+W_2\subset U_1\cap U_2$. Since H is right bounded, there exist open neighborhoods $V_1\ni 0$, $V_i\cdot H\subset W_i$, $i=1,2$. Now

$$\begin{aligned}(V_1\times V_2)\cdot(H\times H)&=\{(v_1,v_2)(h_1,h_2),\ v_i\in V_i,\ h_i\in H,\ i=1,2\}\\&=\{(v_1h_1+v_2h_2,\ v_2h_1+v_1h_2)\}\\&=\{(v_1h_1,\ v_2h_1)+(v_2h_1,\ v_1h_2)\}\\&\subseteqq\{(w_1,w_2)+(w_2',w_1')\}\\&=\{(w_1+w_2',\ w_2+w_1')\}\\&\subseteqq U_1\times U_2.\end{aligned}$$

Hence

$$\begin{aligned}(V_1\times V_2+\Delta)\cdot(H\times H)&=(V_1\times V_2)\cdot(H\times H)+\Delta\cdot(H\times H)\\&\subseteq U_1\times U_2+\Delta,\end{aligned}$$

for Δ is a two-sided ideal in $H\times H$. Let $T=\nu(V_1\times V_2+\Delta)$. Since ν is an open mapping, we have that T is an open neighborhood of 0 in R. Also, $T\cdot R\subseteq V$ and R is right bounded.

Definition 6. A semi-simple halfring H is said to be strongly semi-simple, if the ring R generated by H is also semi-simple.

As a result of Theorem 1 and Lemma 3, we obtain

THEOREM 2. *A locally compact bounded strongly semi-simple halfring H with property F is embeddable in a locally compact bounded semi-simple ring.*

I. Kaplansky [4] has given structure theorems for locally compact rings. He proved that *a locally compact bounded semi-simple ring is the direct sum of a compact semi-simple ring and a discrete semi-simple ring* [4]. Thus the problem of the structure of a locally compact bounded semi-simple ring is reduced to the problem of the structure of a discrete semi-simple ring. We should like a similar reduction in the case of an embeddable locally compact bounded strongly semi-simple halfring, since the structure of a compact semi-simple halfring is known [2]. However, such a splitting is not valid in general and the problem of under what conditions such a phenomenon occurs remains as yet unsolved.

References

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