47. Countable Compactness and Quasi-uniform Convergence

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In his paper [2], R. W. Bagley has given some characterisation of pseudo-compact spaces. In his paper, it is shown that properties of convergence of sequences of continuous function are important. As stated in my paper [4] and Z. Frolík's paper [3], for characterisations of weakly compact spaces, properties of convergence of sequences of quasi-continuous functions are essential. In this note, we shall show that some types of convergence of sequence of upper semi-continuous functions are available for characterisation of countably compact space. One of such an observation was already given by A. Appert [1, p. 102].

Now, let $\{f_n(x)\}$ be a convergent sequence on S, and let $f(x)$ be its limit. $f_n(x)$ is said to be *simply-uniformly* convergent at a point x_0 to $f(x)$, if, for every positive ε and index N, there are an index $n\,(\geq N)$ and a neighbourhood U of x such that $|f_n(x)-f(x)|<\varepsilon$ for x of U. If $f_n(x)$ is simply uniformly convergent to $f(x)$ at every point of S, we say that $f_n(x)$ is *simply uniformly* convergent to $f(x)$, and we shall denote it by $f_n\to f(SU)$. $f_n(x)$ is said to converge to $f(x)$ *quasi-uniformly* on S (in symbol $f_n\to f(QU)$), if, for every $\varepsilon>0$ and N, there is a finite number of indices $n_1, n_2,\cdots, n_k\geq N$ such that for each x at least one of the following relations holds:

$$|f_{n_i}(x)-f(x)|<\varepsilon \qquad (i=1, 2,\cdots, k).$$

Then we shall prove the following

Theorem. *A topological space S is countably compact, if and only if $f_n\to 0$ implies $f_n\to 0$ (QU), where $f_n\in C_+(S)$, and non-negative.*

Proof. Let S be countably compact, suppose that $f_n\to 0$ and $f_n\in C_+(S)$. For a given $\varepsilon>0$, and a given index N, let

$$O_n=\{x\,|\,f_n(x)<\varepsilon\},$$

where $n\geq N$. Since each function $f_n(x)$ is upper semi-continuous, $\{O_n\}_{n=N, N+1,\cdots}$ is open set. $f_n\to 0$ implies that the family $\{O_n\}_{n=N, N+1,\cdots}$ is a countable open covering of S.

Therefore, we can take a finite number of $O_{n_1}, O_{n_2},\cdots, O_{n_k}$ $(n_i\geq N)$ such that $\bigcup_{i=1}^{k} O_{n_i}=S$. Hence for $x\in S$, there is an index n_i $(1\leq i\leq k)$ such that

$$0\leq f_{n_i}(x)<\varepsilon.$$

This shows $f_n\to 0$ (QU).

Conversely, suppose that S is not countably compact, there is a sequence $\{x_n\}$ such that the set $\{x_n\}$ is an infinite isolated set. We shall define $f_n(x)$ as follows:

$$f_n(x)=\begin{cases}1 & \text{for } x=x_m \ (m\geq n)\\ 0 & \text{for } S-\{x_1,\cdots,x_{n-1}\},\end{cases}$$

where $n=1, 2,\cdots$. Then $f_n(x)$ coverges to 0 pointwisely. For a given index N, let us take any $n_1,\cdots,n_k\geq N$, then, for x_m $(m>n_1,\cdots,n_k)\in S$, we have $f_{n_i}(x_m)=1$ $(i=1, 2,\cdots, k)$. This shows that $f_m(x)$ does not quasi-uniformly converge to 0. Hence the proof is complete.

As other characterisation of countably compact space, we have the following

Theorem. The following statements are equivalent:

1) *A topological space S is countably compact.*

2) *For every $f\in C_+(S)$, $f(x)$ takes maximal value at a point of S.*

3) *For any decreasing sequence $\{f_n\}$ of the $C_+(S)$, if $f_n\to 0$, its convergence is uniform.*

4) *For every sequence $\{f_n\}$ of $C_+(S)$, $f_n\downarrow 0$ implies: $f_n(x)$ is μ-convergent to* 0.

A similar result for weakly compact spaces is also true.

Theorem. For any topological space, the following propositions are equivalent:

1) *S is weakly compact.*

2) *For every $f\in C_+(S)\cap Q$-(S), $f(x)$ is bounded from above.*

3) *For every $f\in C_+(S)\cap Q$-(S), $f(x)$ takes the maximal value at a point of S.*

4) *For every $f_n\in C_+(S)\cap Q$-(S), $f_n\downarrow 0$ implies $f_n\Rightarrow 0$.*

5) *For every $f_n\in C_+(S)\cap Q$-(S), $f_n\downarrow 0$ implies: $f_n(x)$ is μ-convergent to* 0.

6) *For every $f_n\in C_+(S)\cap Q$-(S), $f_n\to 0$ implies $f_n\to 0$ (QU).*

Q-(S) is the set of all lower quasi-continuous functions on S, and $C_+(S)$ is the set of all upper semi-continuous functions on S.

References

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[4] K. Iséki: Generalisations of the concept of compactness (in preparation, especially, §1 and §3).