46. On Stable Functional Cohomology Operations

By Nobuo SHIMADA
Mathematical Institute, Nagoya University
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As is well known, the functional primary cohomology operations[1] are inevitably related to the secondary cohomology operations.[2] Recently J. F. Adams[3] has given an axiomatic characterization for stable secondary operations with its important applications. It seems, then, natural and useful indeed to give a similar axiomatic formulation for stable functional operations, and it is our objective.

We follow Adams' notations.[4] Let p be a prime; let A be the Steenrod algebra[5] over Z_p. An A-module is to be a graded left module over the graded algebra A. Let us write $H^*(X)$ for $\sum_q H^q(X, Z_p)$ and $H^+(X)$ for $\sum_{q>0} H^q(X, Z_p)$; then they are A-modules.

Let C_0, C_1 be free A-modules of locally finite type such that $(C_i)_q=0$ if $q<i$ $(i=0, 1)$. Let (d, v) be a pair of an A-map $d: C_1 \to C_0$ of degree zero and a homogeneous element v of C_1. We call φ a stable functional primary cohomology operation associated with (d, v), if it satisfies the following axioms.

AXIOM 1. $\varphi(f, \varepsilon)$ is defined for each pair of a map $f: Y \to X$ and an A-map $\varepsilon: C_0 \to H^+(X)$ of degree $m \geqq 1$ such that $f^*\varepsilon=0$ and $\varepsilon d=0$.

Such a map ε is determined by its values on the elements of an A-base of C_0. It therefore corresponds to a set of elements of $H^+(X)$. In particular, if C_0, C_1 are free on one given generator e_i $(i=0, 1)$ respectively and $de_1=ae_0$ $(a \in A)$, then we write $u=\varepsilon e_0$ and $\varepsilon d=0$ means $au=0$; we may thus consider the operation φ associated with (d, e_1) as a function of one variable u for a fixed map f, where u runs over a subset of $H^+(X)$. In this case we write $a_f(u)$ for $\varphi(f, \varepsilon)$ as usual.

For the next axiom, set $\deg(v)=\nu$, let $\lambda: C_0 \to H^+(Y)$ run over the A-maps of degree $m-1$, and let $L^{m+\nu-1}(d, v; f)$ be the set of elements of the form $\lambda dv + f^*x$ $(x \in H^{m+\nu-1}(X))$.

1) N. E. Steenrod: Cohomology invariants of mappings, Ann. Math., **50**, 954-988 (1949); F. P. Peterson: Functional cohomology operations, Trans. A. M. S., **86**, 197-211 (1957).

2) J. Adem: The iteration of the Steenrod squares in algebraic topology, Proc. Nat. Acad. Sci. U. S. A., **38**, 720-726 (1952); F. P. Peterson and N. Stein: Secondary cohomology operations: two formulas, Amer. J. M., **81**, 281-305 (1959).

3) J. F. Adams: On the nonexistence of elements of Hopf invariant one, Bull. A. M. S., **64**, 279-282 (1958).

4) Loc. cit., 3).

5) H. Cartan: Sur l'itération des opérations de Steenrod, Comment. Math. Helv., **29**, 40-58 (1955).

AXIOM 2. $\varphi(f, \varepsilon) \in H^{m+\nu-1}(Y)/L^{m+\nu-1}(d, v; f)$.

For the next axiom, consider the following (homotopy commutative) diagram.

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle \bar{g}}\Big\uparrow & & \Big\uparrow{\scriptstyle g} \\ Y' & \xrightarrow{f'} & X' \end{array}$$

AXIOM 3. $\varphi(f', g^*\varepsilon) = \bar{g}^*\varphi(f, \varepsilon)$.

For the next axiom, let (X, Y) be a pair, and let $\varepsilon: C_0 \to H^+(X)$ be a map of degree $m \geqq 1$ such that $\varepsilon d = 0$ and $i^*\varepsilon = 0$. We can form the following diagram.

$$\begin{array}{ccccccccc} H^+(Y) & \xleftarrow{i^*} & H^+(X) & \xleftarrow{j^*} & H^+(X, Y) & \xleftarrow{\delta} & H^+(Y) & \xleftarrow{i^*} & H^+(X) \\ & & & {\scriptstyle \varepsilon}\nwarrow & \Big\uparrow{\scriptstyle \eta} & & \Big\uparrow{\scriptstyle \zeta} & & \\ & & & & C_0 & \xleftarrow{(-1)^m d} & C_1 & & \end{array}$$

AXIOM 4. $\varphi(i, \varepsilon) = \{\zeta(v)\} \bmod L^{m+\nu-1}(d, v; i)$.

For the next axiom, let SX be the suspension of X; let $Sf: SY \to SX$ be the suspension of $f: Y \to X$, and let $\sigma: H^+(X) \to H^+(SX)$ be the suspension isomorphism.

AXIOM 5. $\sigma\varphi(f, \varepsilon) = \varphi(Sf, \sigma\varepsilon)$.

Theorem 1. *Given any pair (d, v) as above, there is one stable functional cohomology operation φ associated with it (in the sense of the axioms above), and it is uniquely determined.*

This theorem is proved by the method of the universal example.

The next theorem corresponds to Theorem 3 of Adams.[6]

Theorem 2. a) *Suppose given d, elements v_t in C_1 and operations φ_t associated with the pairs (d, v_t). Suppose $v = \sum_t a_t v_t$ $(a_t \in A)$. Then, we have*

$$\{\varphi(f, \varepsilon)\} = \{\textstyle\sum_t a_t \varphi_t(f, \varepsilon)\} \bmod \textstyle\sum_t a_t L^{m+\nu_t-1}(d, v_t; f) + \operatorname{Im} f^*$$

for the operation φ associated with (d, v).

b) *Suppose given a diagram*

$$\begin{array}{ccc} C_1 & \xrightarrow{m_1} & C_1' \\ {\scriptstyle d}\Big\downarrow & & \Big\downarrow{\scriptstyle d'} \\ C_0 & \xrightarrow{m_0} & C_0' \end{array}$$

in which d, d' are as above, and m_0, m_1 are A-maps of degree zero. Let φ be the operation associated with a pair (d, v) and let φ' be the operation associated with $(d', m_1 v)$. Then we have

$$\varphi(f, \varepsilon' m_0) = \{\varphi'(f, \varepsilon')\}$$

where $\varepsilon': C_0' \to H^+(X)$ is of the sort considered above.

6) Loc. cit., 3).

One may generalize the two formulas of Peterson and Stein[7] as follows.

Theorem 3. a) *Given d as above, element $z=\sum_t a_t e_t$ $(a_t \in A)$ in* $\operatorname{Ker} d$, *where the elements e_t form an A-base of C_1. Let Φ be a stable secondary cohomology operation associated with (d, z) (in the sense of Adams) and let φ_t be the functional operations associated with (d, e_t). Then*

$$f^*\Phi(\varepsilon)=\sum_t a_t\varphi_t(f, \varepsilon) \bmod \sum_t a_t L^{m+\nu_t-1}(d, e_t; f)$$

for each $\varepsilon: C_0 \to H^+(X)$ of degree $m \geqq 1$ and each $f: Y \to X$ such that $\varepsilon d=0$ and $f^*\varepsilon=0$.

b) *Given d as above, element z in* $\operatorname{Ker} d$. *Let $\bar{C}$ be a free A-module generated by one generator $\bar{e}$, let $\bar{d}: \bar{C} \to C_1$ be an A-map of degree zero such that $\bar{d}\bar{e}=z$ and let $\bar{\varphi}$ be the functional operation associated with $(\bar{d}, \bar{e})$. Then there is a secondary operation Φ associated with (d, z) such that*

$$\Phi(f^*\varepsilon)=-\bar{\varphi}(f, \varepsilon d)^{8)} \bmod L^{m+\nu-1}(\bar{d}, \bar{e}; f)$$

for each $\varepsilon: C_0 \to H^+(X)$ and each $f: Y \to X$ such that $f^*\varepsilon d=0$.

In the following we shall show some examples.

Let X be the CW-complex of the form $(S^m \vee S^{m+1}) \smile e^{m+2}$, where e^{m+2} is attached to $S^m \vee S^{m+1}$ by a map $S^{m+1} \to S^m \vee S^{m+1}$ of type $(\eta_m, 2\iota_{m+1})$. Let Y be another S^{m+1}, let $f: Y \to X$ be a map of type $(\eta_m, 0)$. Let u and v be generators of $H^m(X, Z_2)$ and $H^{m+1}(X, Z_2)$ respectively and let y be a generator of $H^{m+1}(Y, Z_2)$. Then we have

$$\overset{2}{Sq}u+\overset{1}{Sq}v=0, \quad f^*u=0 \quad \text{and} \quad f^*v=0.$$

In this situation, let C_0 be as above free on two generators e_i of degree i $(i=0, 1)$, let C_1 be free on one generator $\bar{e}$ of degree 2 and let $d: C_1 \to C_0$ be such that $d\bar{e}=\overset{2}{Sq}e_0+\overset{1}{Sq}e_1$. Then, for the operation φ associated with $(d, \bar{e})$, we have

$$\varphi(f, \varepsilon)=y \qquad \bmod \text{zero}$$

for $\varepsilon: C_0 \to H^+(X)$ such that $\varepsilon e_0=u$ and $\varepsilon e_1=v$. This will give the most simple example of non-trivial stable functional operations of two variables.

For the next example, let C_0 be free on two generators e_i of degree i $(i=0, 2)$, let C_1 be free on two generators $\bar{e}_2$ and $\bar{e}_3$ of degrees 2 and 3 respectively and let $d: C_1 \to C_0$ be such that $d\bar{e}_2=\overset{2}{Sq}e_0$ and $d\bar{e}_3=\overset{2}{Sq}\overset{1}{Sq}e_0$ $+\overset{1}{Sq}e_2$. Then $z=\overset{2}{Sq}\bar{e}_2+\overset{1}{Sq}\bar{e}_3$ is in $\operatorname{Ker} d$.

Theorem 4. *Take d, z as above. Then there is uniquely the second-*

7) Loc. cit., 2).

8) The involved sign is caused by the anticommutativity of a certain diagram corresponding that appeared in the proof of Lemma 6.2. of Peterson and Stein, loc. cit., 2). Cf. also Y. Nomura: On mapping sequences (to appear in Nagoya Mathematical Journal).

ary cohomology operation Φ *associated with* (d, z) *such that*

$$\varphi^2(u)=\{\Phi(u, v)\} \qquad \text{mod } Q$$

for classes $u \in H^m(X, Z_2)$ *and* $v \in H^{m+2}(X, Z_2)$ *with relations* $\overset{2}{Sq}u=0$, $\overset{2}{Sq}\overset{1}{Sq}u+\overset{1}{Sq}v=0$ *and for a certain subgroup* Q, *where* φ^2 *is the operation of Chow*[9] *and* $\Phi(u, v)$ *denotes* $\Phi(\varepsilon)$ *for* $\varepsilon: C_0 \to H^+(X)$ *such that* $\varepsilon e_0=u$ *and* $\varepsilon e_2=v$.

There is another similar case and these examples show that certain binary secondary operations in the sense of Adams[10] are related to the unary operations φ^i of Chow. In the proof of the above theorem, we make use of Theorem 3, a) above and some calculations of the involved functional operations.

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9) S. Chow: Steenrod's operations and homotopy groups (II), Acta Mathematica Sinica (in Chinese), **9**, 243-263 (1959).

10) Contenting with less axiomatic but satisfactory conceptual definitions for stable secondary or functional cohomology operations, one could go further. The operations φ^i $(i \geqq 1)$ originally defined by cochain formulas are thus reduced to certain (in general, binary) operations in a generalized sense as above.