45. *Extended Non-Euclidean Geometry*

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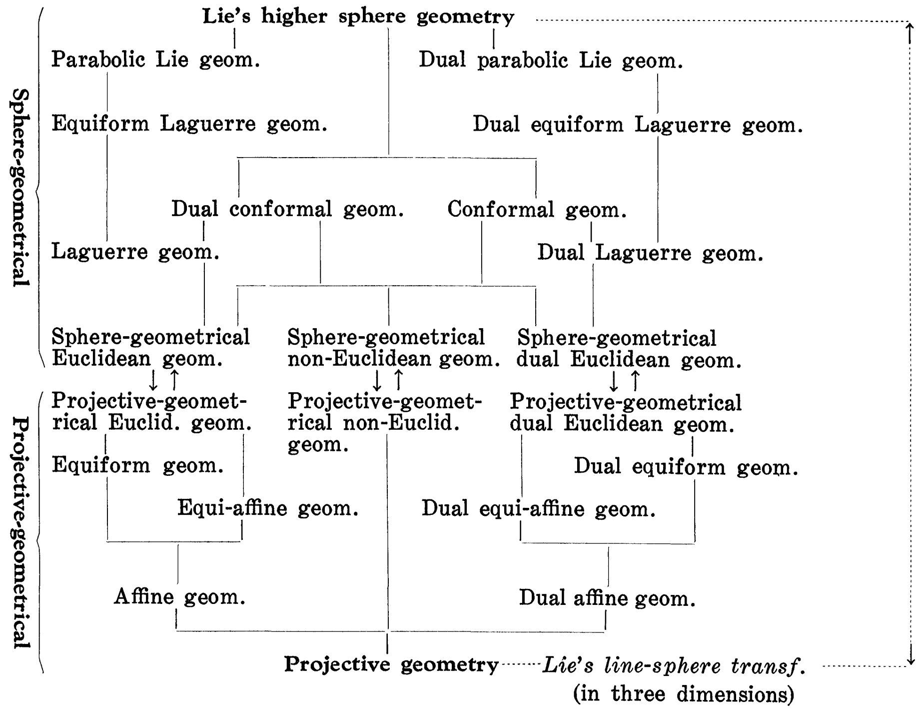
In [1–4], I have started to extend all the branches of geometry of the following table by extending the respective transformation group parameters to functions of coordinates:



In this note an *extended non-Euclidean geometry* will be established. It should be noticed that the extensions of the so-called

Cayley-Klein | Poincaré-Klein

representation are unified in it by mapping onto each other by an *extended Darboux-Liebmann transformation,* which is an extended equiform transformation [3].

The extended non-Euclidean geometry so obtained is realized in the differentiable manifolds (atlas) in the sense of S. S. Chern and C. Ehresmann.

1. Extended projective geometry. I have established [4] an *extended equi-affine group* of transformations

(1.1) $\bar{\xi}^l = a^l_m(\xi^p)\xi^m + a^l_0, \quad (|a^l_m(\xi^p)|=1,\ a^l_0=\text{const.},\ l, m, \cdots=1, 2, \cdots, n),$

(1.2) $$d\bar{\xi}^l = a^l_m(\xi^p)d\xi^m,$$

where ξ^l and $\bar{\xi}^l$ are II-geodesic parallel coordinates and introduced the extended affine length s of curves by

(1.3) $$ds = |d\xi d^2\xi \cdots d^n\xi|^{2/n(n+1)}.$$

The ordinary parallel coordinates x^l are special kinds of ξ^l.

By the inverse transformation

(1.4) $$x^l = \Omega^l_m(\xi^p)\xi^m + \Omega^l_0, \quad (|\Omega^l_m(\xi^p)| = 1)$$

of the extended equi-affine transformation (1.1), the ordinary hyperplane

(1.5) $$c_l x^l + c_0 = 0$$

is transformed into the figure

(1.6) $$c_l \Omega^l_m(\xi^p)\xi^m + (c_l \Omega^l_0 + c_0) = 0.$$

Writing

(1.7) $$\alpha_m(\xi^p) \overset{\text{def}}{=} c_l \Omega^l_m(\xi^p), \quad \alpha_0 \overset{\text{def}}{=} c_l \Omega^l_0 + c_0,$$

for (1.6), we have

(1.8) $$\alpha_m \xi^m + \alpha_0 = 0.$$

Taking $(n+1)$ II-geodesic $(n-1)$-flats

(1.9) $$\alpha^i_m \xi^m + \alpha^i_0 = 0, \quad (i = 1, 2 \cdots, n+1),$$

we set

(1.10) $$\rho(\xi^p) \cdot X^i \overset{\text{def}}{=} \alpha^i_m \xi^m + \alpha^i_0, \quad (\rho(\xi^p) \neq 0).$$

The ratios

(1.11) $$X^1 : X^2 : \cdots : X^n : X^{n+1}$$

define a point in the equi-affine space A^n.

In order that II-geodesic curves $d^2\xi^l/ds^2 = 0$ may be transformed into II-geodesic curves $d^2(\rho \cdot X^i)/ds^2 = 0$, since $d^2(\rho \cdot X^i)/ds^2 = \alpha^i_m d^2\xi^m/ds^2$, we must have

(1.12) $$\rho \cdot X^i = a^i s + c^i, \quad (i = 1, 2, \cdots, n+1),$$

where a^i and c^i are constants. Thus we have

(1.13) $$X^i : X^j = (a^i s + c^i)/(a^j s + c^j), \quad (a^i c^j - a^j c^i = 1;\ i, j = 1, 2, \cdots, n+1;\ i \neq j).$$

We will call the ratios (1.13) the II-*geodesic projective point coordinates.*

The (1.10) may be rewritten:

(1.14) $$\rho \cdot X^i = \alpha^i_m a^m_l(x^p)x^l + \alpha^i_0, \quad (|a^m_l(x^p)| = 1),$$

which we will call an *extended projective transformation.* By (1.2), there exists $d\xi^l$ in A^{n+1} such that

$$d\xi^i = \alpha^i_m a^m_q(x^p)dx^q + \alpha^i_0 d1.$$

Hence we may identify $d\xi^i$ with $d(\rho \cdot X^i)$:

(1.15) $$d(\rho \cdot X^i) = d\xi^i.$$

Similarly, for $\alpha^i_j(X^k)dX^j$, $(|\alpha^i_j(X^k)| = 1)$, we have

(1.16) $$d\bar{\xi}^i = \alpha^i_j(X^k)dX^j, \quad (\rho \neq 0;\ j, k = 1, 2, \cdots, n+1),$$

whence, quite as in the case of (1.1), we have

(1.17) $$\bar{\xi}^i=\alpha^i_j(X^k)X^j+\alpha^i_0,\ (\alpha^i_0=\text{const.}),$$
so that

(1.18) $$\rho\cdot\bar{X}^i=\alpha^i_j(X^k)X^j,\ (\rho\neq 0;\ |\alpha^i_j(X^k)|=1;\ \rho\cdot\bar{X}^i=\bar{\xi}^i-\alpha^i_0).$$

In the case $n=1$, by (1.12), we have

(1.19) $$\bar{s}=(a^1s+c^1)/(a^2s+c^2),\ (a^1c^2-a^2c^1=1),$$
where we have put $\cdot\,\bar{s}=X^1/X^2$. Thus we may interpret s and $\bar{s}$ as *double ratio coordinates on the* II-*geodesic curves.*

We will call (1.18) an *extended projective transformation.* The totality of the extended projective transformations forms a group, which we will call the *extended projective group.* It contains the ordinary projective group as a subgroup. The geometry under the extended projective group will be called the *extended projective geometry.*

2. Extended Cayley-Klein representation. A II-*geodesic hyperquadric*

(2.1) $$X^iX^i=0$$
will be called the *Absolute.*

The ratios of u_i such that

(2.2) $$\sigma\cdot u_i=X^i,\ (\sigma\neq 0),$$
will be called the *projective* II-*geodesic* $(n-1)$-*flat coordinates.* (2.1) becomes

(2.3) $$u_iu_i=0.$$

The II-geodesic projective coordinates (X^i) and (u_i) shall be normalized as follows:

(2.4) $$X^iX^i=k^2. \qquad\qquad u_iu_i=1.$$

The extended projective transformations $\alpha^i_j(X^k)$ will be called the *extended non-Euclidean transformations.* The totality of them forms a group, which we will call the *extended non-Euclidean group.* An *extended non-Euclidean geometry* belongs to it.

The *distance* d	The *angle* ϕ
between two	
points $X^i, X^{i'}$	II-geodesic $(n-1)$-flats u_i, u'_i
is given by	
(2.5) $\cos\dfrac{d}{k}=\dfrac{X^iX^{i'}}{\sqrt{X^iX^i}\,\sqrt{X^{i'}X^{i'}}}$.	$\cos\phi=\dfrac{u_iu'_i}{\sqrt{u_iu_i}\,\sqrt{u'_iu'_i}}$.

All the theorems and theories in the classical non-Euclidean geometry are retained under the extended non-Euclidean group.

3. Extended Poincaré-Klein representation. Take a II-geodesic hypersphere

(3.1) $$\xi^p\xi^p=k^2,\ (p=1,2,\cdots,n),$$
where ξ^p are the II-geodesic rectangular coordinates [3] and name it

the *Absolute*.

Apply the extended equiform transformation [2]

(3.2) $$\bar{\xi}^p = 2k^2\xi^p/(\xi^p\xi^p + k^2)$$

to the II-geodesic $(n-1)$-flat

(3.3) $$u_p\bar{\xi}^p + u_{n+1} = 0.$$

Then the II-geodesic hypersphere

(3.4) $$2k^2(u_p\xi^p) + u_{n+1}(\xi^p\xi^p + k^2) = 0,$$

which meets (3.1) orthogonally, is obtained. In this way we obtain an *extended Poincaré-Klein representation of the extended non-Euclidean space.*

The (3.4) may be rewritten:

(3.5) $$2k^2(u_p\xi^p + u_{n+1}) + u_{n+1}(\xi^p\xi^p - k^2) = 0.$$

4. An extended Darboux-Liebmann transformation. Considering the II-geodesic hypersphere

(4.1) $$\bar{\xi}^p\bar{\xi}^p = k^2$$

as *Absolute* and the II-geodesic $(n-1)$-flat

(4.2) $$u_p\bar{\xi}^p + u_{n+1} = 0$$

as the II-geodesic $(n-1)$-flat of the extended non-Euclidean space, we obtain an *extended Cayley-Klein representation* $\bar{P}(\bar{\xi}^p)$.

Considering the II-geodesic hypersphere

(4.3) $$\xi^p\xi^p = k^2$$

as *Absolute* and the II-geodesic hypersphere

(4.4) $$2k^2(u_p\xi^p + u_{n+1}) + u_{n+1}(\xi^p\xi^p - k^2) = 0$$

as a II-geodesic $(n-1)$-flat of the extended non-Euclidean space, we obtain an *extended Poincaré-Klein representation* $P(\xi^p)$.

The (4.4) is the map of the (4.2) by (3.2) and is a II-*geodesic* $(n-1)$-*flat.*

The transformation (3.2) is a mutual mapping of the extended

Cayley-Klein | Poincaré-Klein

representation and will be called the *extended Darboux-Liebmann transformation.*

References

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