64. A Galois Theory for Finite Factors

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According to a closed analogy between theories of classical simple algebras and continuous finite factors, it is natural to ask that a continuous finite factor obeys a kind of Galois theory. Literally, it is known that I. M. Singer [4] gave an attempt in this direction.

This note will present a trial towards it in the following

THEOREM. *If A is a continuous finite factor acting standardly on a separable Hilbert space H, if G is a finite group of outer automorphisms of A, if B is the subfactor of A consisting of all elements invariant under G, and if moreover the commutor B' of B is finite. Then, the lattices of all subgroups of G and of all intermediate subfactors between B to A are dually isomorphic under the Galois correspondence which carries a subgroup F to an intermediate subfactor C invariant under F in element-wise.*

It is expected that the assumption on B' is provable from the finiteness of G for which the authors hope to discuss in the next occasion. It is also to be remarked that the continuity assumption on A in the theorem is superfluous since a discrete finite factor has no non-trivial group of outer automorphisms.

1. Since A acts standardly on H, there is a unitary u_g for any g such as

(1) $$x^g=u_g x u_g^*,$$

where x^g means the action of g on $x\in A$. Throughout the remainder, for the sake of convenience, it is to be assumed that the correspondence $g\rightarrow u_g$ satisfies

(2) $$u_{g^{-1}}=u_g^*.$$

It is to be noticed that u_g belongs to B', since $x=x^g=u_g x u_g^*$ by the assumption.

LEMMA 1. *By* (1), *g gives an outer automorphism on A'.*

If $x\in A'$, then for any $a\in A$, (1) and (2) imply

$$ax^g=au_g x u_g^*=u_g a^{g^{-1}} x u_g^*=u_g x a^{g^{-1}} u_g^*=u_g x u_g^* a=x^g a,$$

which shows that g conserves A'. Hence (1) gives an automorphism on A'. If it is inner, then there is a unitary $w\in A'$ such that $x^g=w^*xw$ or $u_g x u_g^*=w^*xw$ for any $x\in A'$, whence $wu_g x=xwu_g$ for any $x\in A'$, that is, wu_g commutes with every element of A'. Hence the unitary operator $w'=wu_g$ belongs to A. Therefore, by $w\in A'$,

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$$x^g = u_g x u_g^* = w^* w' x w'^* w = w' x w'^*,$$

which is a contradiction since g was not inner in A.

LEMMA 2. *In B', u_g is orthogonal to A'.*

Since B' is a finite factor, by the well-known technique due to Dixmier [1] and Umegaki [5], there exists the conditional expectation ϵ conditioned by A' which projects B' onto A'. By (1), $x^g u_g^\epsilon = u_g^\epsilon x$ for any $x \in A'$, whence $u_g^\epsilon = 0$ by [2, Lemma 1], that is, u_g is orthogonal to A'. This proves the lemma.

LEMMA 3. *$\{u_g \mid g \in G\}$ are linearly independent over A'.*

We shall show that $A'u_g$ is orthogonal to $A'u_h$ for $g \neq h$ considering B' as a prehilbert space with respect to the usual inner product introduced by the trace τ. For any $x \in A$,

$$u_g u_h^* x u_h u_g^* = x^{h^{-1}g} = u_{h^{-1}g} x u_{h^{-1}g}^*,$$

whence $vx = xv$ or $v \in A'$ where $v = u_{h^{-1}g}^* u_g u_h^*$. Therefore, putting $c = a u_{h^{-1}g} v b^* u_{h^{-1}g}^*$ for $a, b \in A'$, $u_g u_h^* = u_{h^{-1}g} v$ implies

$$\tau(a u_g u_h^* b^*) = \tau(a u_{h^{-1}g} v b^*) = \tau(a u_{h^{-1}g} v b^* u_{h^{-1}g}^* u_{h^{-1}g}) = \tau(c u_{h^{-1}g}).$$

Since c belongs to A' by the definition, Lemma 2 implies that $\tau(a u_g u_h^* b^*) = 0$ unless $g = h$, which proves the lemma.

LEMMA 4. *Each element b of B' is uniquely expressible as*

$$(3) \qquad b = a_1 + a_g u_g + \cdots + a_k u_k,$$

where $a_1, a_g, \cdots, a_k$ are elements of A'.

By virtue of Lemma 3, it is sufficient to show that $\{u_g \mid G\}$ and A' generate B'. Let C be the totality of all forms satisfying (3). Clearly C is metrically closed in B' and a subalgebra of B', whence C is a von Neumann subalgebra of B'. Taking the commutors, we have $A \geqq C' \geqq B$. Now, each element $c \in C'$ is permutable with each u_g, whence c is invariant under G, that is, c belongs to B or $C = B'$, which proves the lemma.

2. The method here employed is essentially the same as that of [2]. It is to be expected that B' becomes the crossed product with respect to a suitable factor set studied in [3] for which the authors hope to discuss in the future.

LEMMA 5. *An intermediate von Neumann subalgebra D between A' and B' is spanned by A' and a subgroup F of G.*

Let $\div$ be the conditional expectation (in B') conditioned by D. For any $k \in G$, by Lemma 4,

$$u_k^\div = a_1 + a_g u_g + \cdots + a_h u_h.$$

Since $u_k^\div x = x^k u_k^\div$ for any $x \in A'$, Lemma 4 again yields $a_g x^g u_g = x^k a_g u_g$ or $a_g x^g = x^k a_g$ for any $x \in A'$. Thus, $a_g^{g^{-1}} x = x^{kg^{-1}} a_g^{g^{-1}}$ for any $x \in A'$. Consequently, by [2, Lemma 1], $a_g = 0$ unless $g = k$. This implies either $u_k^\div = u_k$ or $u_k^\div = 0$, or in other words, either $u_k \in D$ or u_k being orthogonal to D. Clearly, $F = \{k \mid u_k \in D\}$ is a subgroup of G. Therefore, D is obviously generated by A' and F.

LEMMA 6. *$A \frown B'$ is the scalar multiples of the identity.*

If b is an element of $A \frown B'$, then (3) implies $a_1 a = a a_1$ and $a_g a^g = a a_g$ for any $g \neq 1$, comparing the coefficients of ba and ab for arbitrary $a \in A'$. The first implies that a_1 is a scalar since A' is a factor, and the second implies by [2, Lemma 1] that a_g vanishes for each $g \neq 1$, which show the lemma by (3).

COROLLARY. *Each intermediate von Neumann subalgebra between B and A is a subfactor of A. Consequently, each intermediate von Neumann subalgebra between A' and B' is a subfactor of B', too.*

If D is an intermediate von Neumann subalgebra between B and A, then Lemma 6 and $D \frown D' \leqq D \frown B' \leqq A \frown B'$ show the first half, since $D \leqq A$ and $D' \leqq B'$ by the hypothesis. The remainder is now obvious.

LEMMA 7. *The lattices of subgroups of G and of intermediate subfactors between A' and B' are isomorphic by the natural correspondence.*

This is clear by Lemmas 5-6 and Corollary.

LEMMA 8. *The lattices of subgroups of G and of intermediate subfactors between A and B are dually isomorphic.*

Taking the commutor, the lattice of all intermediate subfactors between A and B is naturally dually isomorphic to that of all subfactors between B' and A', whence the lemma follows from Lemma 7.

3. Now, the theorem is nearly obvious. For any subgroup F of G, the correspondences of Lemmas 7 and 8 carry F to a subfactor C whose commutor C' is spanned by A' and F, whence each element c of C commutes with u_g for each $g \in F$, and so C is invariant under F in element-wise. Conversely, an intermediate subfactor C, which is invariant under a subgroup F in element-wise, corresponds naturally to F. This proves the theorem.

References

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