63. On the Semi-exact Canonical Differentials of the First Kind

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(Comm. by K. KUNUGI, M.J.A., May 19, 1960)

1. **Introduction.** Let R be an arbitrary open Riemann surface of finite genus g. We shall denote by $\mathfrak{K}$ the class of semi-exact canonical differentials[1] (or integrals of these) on R, and by δ an arbitrary divisor of finite order $d[\delta]$ on R. Then, with differentials and integrals (functions) of $\mathfrak{K}$, the following Riemann-Roch's theorem was established by Prof. Kusunoki:[2]

$$A[\delta^{-1}]-B[\delta]=2(d[\delta]-g+1),$$

where $A[\delta^{-1}]$ denotes the number of linearly independent (in the real sense) functions $\in\mathfrak{K}$, which are single-valued on R and multiples of δ^{-1}, and $B[\delta]$ the number of linearly independent differentials $\varphi\in\mathfrak{K}$ which are multiples of δ. If we take a divisor $\delta=P^r$ $(0\leqq r\leqq g,\ P\in R)$ we have therefore $B[P^r]\geqq 2(g-r)$, in particular, $B[P^g]\geqq 0$. In the present paper, we shall show that the set of the points where $B[P^g]=0$ is dense in R, and the properties of the points. Theorem 3 shows the existence of parallel slit mappings under an additional condition on the boundaries. And finally, some remarks on curves in R and points lying on the boundaries will be given.

2. **Theorem 1.** *The set of the points at which $B[P^g]=0$ is dense in R.*

Proof. If the theorem is not true, there is an open set U in R which does not contain any point where $B[P^g]=0$. Let P_0 be an arbitrary point in U. In terms of a local parameter $z=\Phi(P)(\Phi(P_0)=0)$ about P_0, each of the $2g$ basis differentials[3] $\varphi_j\in\mathfrak{K}$ $(j=1,2,\cdots,2g)$ of the first kind on R, can be represented as $\varphi_j=f_j(z)dz$, where the $f_j(z)$ $(j=1,2,\cdots,2g)$ are linearly independent analytic functions of $z=x+iy$ about P_0. We consider the following real function which is analytic with respect to x and y:

$$V_{2g}(z)\equiv|R_{2g}^0 I_{2g}^0 R_{2g}^1 I_{2g}^1\cdots R_{2g}^{g-1} I_{2g}^{g-1}|$$

where

1) Cf. Kusunoki [3, pp. 241-242], and Nevanlinna [4].
2) Kusunoki [3, Theorem 8], and Kusunoki [2].
3) The existence of these basis differentials was verified in Kusunoki [3, Theorem 1].

$$R_k^j=\begin{pmatrix}\operatorname{Re} f_1^{(j)}(z)\\ \operatorname{Re} f_2^{(j)}(z)\\ \vdots\\ \operatorname{Re} f_k^{(j)}(z)\end{pmatrix} \quad\text{and}\quad I_k^j=\begin{pmatrix}\operatorname{Im} f_1^{(j)}(z)\\ \operatorname{Im} f_2^{(j)}(z)\\ \vdots\\ \operatorname{Im} f_k^{(j)}(z)\end{pmatrix} \quad \begin{matrix}(j=0,1,\cdots,g-1)\\ (k=1,2,\cdots,2g),\end{matrix}$$

here $f_i^{(j)}$ denotes the j-th derivative of f_i and $f_i^{(j)}$ the function f_i itself. The $V_{2g}(z)$ vanishes at a point P if and only if $B[P^g]>0$. Therefore we see $V_{2g}(z)\equiv 0$ on U. Then, we can conclude that the $f_j(z)$ $(j=1,2,\cdots,2g)$ are linearly dependent, which leads us to a contradiction.

The proof can be done as follows. We shall denote by $V_r(z)$ the determinant which consists of the first r rows and r columuns of the $V_{2g}(z)$. Since $V_2(z)\not\equiv 0$[4] on U, there must be such k, $1<k\leqq g$, for which $V_{2k}(z)\equiv 0$, but $V_{2r}(z)\not\equiv 0$ for every r, $1\leqq r<k$. There may be the following three cases.

1°) $V_{2k}(z)\equiv 0$, $V_{2(k-1)}(z)\not\equiv 0$, $V_{2k-1}(z)\equiv 0$ and $|R_{2k-1}^0 I_{2k-1}^0\cdots R_{2k-1}^{k-2} I_{2k-1}^{k-2} I_{2k-1}^{k-1}|\equiv 0$.

Since $V_{2(k-1)}(z)\not\equiv 0$, we can find an open set $U'(\subset U)$ on which $V_{2(k-1)}(z)\neq 0$. Then, there are $2k-1$ real functions $u_j(z)$ $(j=1,2,\cdots,2k-1)$ which are analytic with respect to x and y such that

$$\begin{aligned}&\sum_{j=1}^{2k-1} u_j(z)\operatorname{Re} f_j^{(r)}(z)=0\\ &\sum_{j=1}^{2k-1} u_j(z)\operatorname{Im} f_j^{(r)}(z)=0\end{aligned}\qquad (r=0,1,\cdots,k-1)$$

are satisfied on U'. Dividing these identities by $u_{2k-1}(z)=V_{2(k-1)}(z)\neq 0$ we get

$$(1)\qquad \begin{aligned}&\sum_{j=1}^{2k-2} v_j(z)\operatorname{Re} f_j^{(r)}(z)+\operatorname{Re} f_{2k-1}(z)=0\\ &\sum_{j=1}^{2k-2} v_j(z)\operatorname{Im} f_j^{(r)}(z)+\operatorname{Im} f_{2k-1}(z)=0\end{aligned}\qquad (r=0,1,\cdots,k-1)$$

where $v_j(z)=u_j(z)/u_{2k-1}(z)$ $(j=1,2,\cdots,2k-2)$. By differentiating (1) with respect to x and using the identities which can be obtained from (1) by putting $r+1$ for r, we obtain

$$\begin{aligned}&\sum_{j=1}^{2k-2} \frac{\partial v_j(z)}{\partial x}\operatorname{Re} f_j^{(r)}(z)=0\\ &\sum_{j=1}^{2k-2} \frac{\partial v_j(z)}{\partial x}\operatorname{Im} f_j^{(r)}(z)=0\end{aligned}\qquad (r=0,1,\cdots,k-2).$$

Since the determinant $V_{2(k-1)}(z)$ of the coefficients of this system of equations is not zero on U', we have $\dfrac{\partial v_j(z)}{\partial x}\equiv 0$, that is, $v_j(z)\equiv c_j$ (real constants) $(j=1,2,\cdots,2k-2)$ on a curve γ, corresponing to $y\equiv$constant. Therefore we have

$$\sum_{j=1}^{2k-2} c_j f_j(z)+f_{2k-1}(z)=0$$

4) Cf. Kusunoki [3, p. 255].

on γ, and so is on U'. This shows that the $f_1, f_2, \cdots, f_{2g}$ are linearly dependent.

2°) $V_{2k}(z)\equiv 0$, $V_{2(k-1)}(z)\not\equiv 0$ and $V_{2k-1}(z)\not\equiv 0$.

By differentiating $V_{2k}(z)\equiv 0$ by x and y respectively, we have

$$(2) \qquad |R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}R^k_{2k}I^{k-1}_{2k}| \equiv -|R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}R^{k-1}_{2k}I^k_{2k}|$$

and

$$(3) \qquad |R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}I^k_{2k}I^{k-1}_{2k}| \equiv |R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}R^{k-1}_{2k}R^k_{2k}|.$$

Forming the derivatives of the second order of $V_{2k}(z)\equiv 0$ with respect to x and y, we can easily verify that

$$(4) \qquad |R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}R^k_{2k}I^k_{2k}|\equiv 0.$$

This shows that there are functions $a_j(z)$ and $b_j(z)$ $(j=0,1,\cdots,k-2;k)$ such that

$$\sum_{r=0}^{k-2}\{a_r(z)R^r_{2k}+b_r(z)I^r_{2k}\}+a_k(z)R^k_{2k}+b_k(z)I^k_{2k}\equiv 0.$$

Now it is proved that the determinants in (3) vanish in U. Indeed, if $a_k(z)\equiv 0$ or $b_k(z)\equiv 0$, they vanish obviously. In case of $a_k(z)\not\equiv 0$ and $b_k(z)\not\equiv 0$, we can represent I^k_{2k} by $R^0_{2k}, I^0_{2k}, \cdots, R^{k-2}_{2k}, I^{k-2}_{2k}$ and R^k_{2k} as

$$(5) \qquad I^k_{2k}\equiv\frac{-1}{b_k(z)}\left[\sum_{r=0}^{k-2}\{a_r(z)R^r_{2k}+b_r(z)I^r_{2k}\}+a_k(z)R^k_{2k}\right]$$

and from (2), (3) and (5), we have

$$\begin{aligned}
&|R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}R^{k-1}_{2k}R^k_{2k}|\\
&\equiv|R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}I^k_{2k}I^{k-1}_{2k}|\\
&\equiv-\frac{a_k(z)}{b_k(z)}|R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}R^k_{2k}I^{k-1}_{2k}|\\
&\equiv\frac{a_k(z)}{b_k(z)}|R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}R^{k-1}_{2k}I^k_{2k}|\\
&\equiv-\left[\frac{a_k(z)}{b_k(z)}\right]^2|R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}R^{k-1}_{2k}R^k_{2k}|
\end{aligned}$$

which shows that the determinant on the left hand side must vanish identically. Since $V_{2k-1}(z)\not\equiv 0$, we have therefore

$$\sum_{j=1}^{2k}u_j(z)\,\mathrm{Re}\,f_j^{(r)}(z)=0 \qquad (r=0,1,\cdots,k)$$

$$\sum_{j=1}^{2k}u_j(z)\,\mathrm{Im}\,f_j^{(r)}(z)=0 \qquad (r=0,1,\cdots,k-1)$$

and the linear dependence of the $f_j(z)$ $(j=1,2,\cdots,2g)$ can be verified quite analogously as in the above-mentioned case.

3°) $V_{2k}(z)\equiv 0$, $V_{2(k-1)}(z)\not\equiv 0$, $V_{2k-1}(z)\equiv 0$ but $|R^0_{2k-1}I^0_{2k-1}\cdots R^{k-2}_{2k-1}I^{k-2}_{2k-1}I^{k-1}_{2k-1}|\not\equiv 0$.

By the same reasoning as in 2°) from (2), (3) and (4), we have

$$|R^0_{2k}I^0_{2k}\cdots R^{k-2}_{2k}I^{k-2}_{2k}I^{k-1}_{2k}I^k_{2k}|\equiv 0$$

and

$$\sum_{j=1}^{2k} u_j(z)\,\mathrm{Re}\,f_j^{(r)}(z)=0 \qquad (r=0, 1, \cdots, k-1)$$

$$\sum_{j=1}^{2k} u_j(z)\,\mathrm{Im}\,f_j^{(r)}(z)=0 \qquad (r=0, 1, \cdots, k)$$

where $u_{2k}(z)\equiv| R_{2k-1}^{0} I_{2k-1}^{0} \cdots R_{2k-1}^{k-2} I_{2k-1}^{k-2} I_{2k-1}^{k-1} | \not\equiv 0$. The linear dependence of the $f_j(z)$ $(j=0, 1, \cdots, 2g)$ can be concluded analogously, q.e.d.

3. According to this theorem, we can always take a point P on R such that $B[P^g]=0$, hence $A[P^{-g}]=2$ by Riemann-Roch's theorem. This means the following

Theorem 2.[5] *For any point P on a dense subset of R, there does not exist any single-valued function $\in \mathfrak{K}$ on R which has a single pole of order at most g at P. At such point $P \in R$ the identities $B[P^r]=2(g-r)$ hold for every r, $0\leqq r\leqq g$. This shows that $B[P^r]-B[P^{r+1}]=2$ for r, $0\leqq r<g$.*

Proof. By Riemann-Roch's theorem, we have $B[P^r]\geqq 2(g-r)$ $(0\leqq r\leqq g)$. Now suppose that there is a number r $(0\leqq r\leqq g)$ such that $B[P^r]>2(g-r)$, then we have $A[P^{-r}]>2$. But this is incompatible with the fact that $A[P^{-g}]=2$, q.e.d.

If we choose a point $P\in R$ such that $B[P^g]=0$, we have $A[P^{-(g+1)}]=4$. This shows by Theorem 2, that there are two linearly independent single-valued functions $\in \mathfrak{K}$ on R, each of which has a single pole of order just $g+1$ at P. On the other hand, when the genus of R is finite and q is the number (counted with multiplicities) of poles of a single-valued function $f\in \mathfrak{K}$ on R, f is at most q-valent on R.[6] Therefore each of these functions gives a conformal mapping of R onto a domain in a $(g+1)$-sheeted covering surface of the sphere. Further if the boundaries of R consist of a finite number of closed Jordan curves, the images of the boundaries by these functions are slits along linear segments which are parallel to the imaginary axis. When we multiply these functions by i, we get conformal mappings of R onto covering surfaces of the plane, which are $(g+1)$-sheeted and have slits parallel to the real axis. Thus the following theorem is established.

Theorem 3. *Suppose that the boundaries of R consist of a finite number of closed Jordan curves, then there exist conformal mappings of R onto covering surfaces of the plane, which are $(g+1)$-sheeted and have slits along parallel segments. We can take a point $P\in R$, as a pole of the functions, arbitrary near the prescribed point, and when we fix a point P at which $B[P^g]=0$ as a pole of the functions, there are two such functions which are linearly independent.*

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5) This theorem corresponds to the classical Weierstrass gap theorem for the points where $B[P^g]=0$. Cf. Springer [5, Theorems 10-18, p. 272].

6) Kusunoki [3, (ii), p. 250].

4. Next, we take any analytic curve γ in the interior of R. If a point P_0 such that $B[P_0^g]=0$ lies on γ, there may exist at most a finite number of points on γ such that $B[P^g]>0$. Otherwise, the set of such points has a cluster point P_1 on γ. Let z be a local parameter about P_1 which brings P_1 to 0 and represents γ by $y\equiv 0$, so the $V_{2g}(z)$ may be considered as a real analytic function of x on the intersection of γ and the parameter neighbourhood of P_1, and it is zero at all points where $B[P^g]>0$. Then, we have $V_{2g}(z)\equiv 0$ on γ. But this is contradictory to the fact that $B[P_0^g]=0$. Therefore we have the following two cases:

i) *There are at most a finite number of points at which* $B[P^g]>0$, $P\in\gamma$.

ii) *At all points* $P\in\gamma$, $B[P^g]>0$.

But we can easily see that *the second case may happen only for a countable number of curves in the interior of* R. In fact, under the consideration of i) and Theorem 1, we can find a neighbourhood about any point $P\in R$, whose boundary curve intersects only a finite number of such curves. And the fact that there are only a finite number of such closed curves in the neighbourhood is easily verified by using i) and Theorem 1.

5. We shall now consider the boundary curves Γ of R. Suppose C is an analytic curve contained in Γ. For every interior point P of C, we can prove that $B[P^r]\geqq 2g-r$ $(0\leqq r\leqq 2g)$. To see this, we choose a local parameter z about P, such that $\Phi(P)=0$ and C corresponds to $y\equiv 0$. Let $\varphi_j\equiv f_j(z)dz$, and we continue $f_j(z)$ as $-\overline{f_j(\bar{z})}$ across C a little outside of R. On C we have $\mathrm{Re}\,\varphi_j\equiv 0$, i.e. $\mathrm{Re}\,f_j(z)\equiv 0$ and $\mathrm{Re}\,f_j^{(k)}(z)=0$ $(j=1,2,\cdots,2g;\ k=1,2,\cdots)$. The number of linearly independent vectors $(c_1, c_2,\cdots,c_{2g})$ with $2g$ real constants which fulfill the system of r equations:

$$\sum_{j=1}^{2g} c_j\,\mathrm{Im}\,f_j^{(k)}(z)=0 \qquad z\in C,\quad (k=0,1,\cdots,r-1)$$

is equal to $B[P^r]$ and this is $\geqq 2g-r$ for r, $0\leqq r\leqq 2g$.

On an analytic curve included in Γ, there does not exist, except at most a finite number of points, any point where $B[P^{2g}]>0$. Indeed, suppose that there exist an infinite number of such points on C, then they have a cluster point, say P_0, on C. We select a suitable local parameter z which represents C by $y\equiv 0$. Then, the following real analytic function

$$\widetilde{V}_{2g}(x)\equiv|\,I_{2g}^0 I_{2g}^1\cdots I_{2g}^{2g-2} I_{2g}^{2g-1}\,|$$

must vanish identically on C. But the $\widetilde{V}_{2g}(x)$ is Wronskian determinant of the functions $\mathrm{Im}\,f_1(x)$, $\mathrm{Im}\,f_2(x),\cdots,\mathrm{Im}\,f_{2g}(x)$, and its vanishing identically implies that they are linearly dependent, which is absurd.

Therefore *for any boundary point* $P \in C$, *we have obtained the inequality*; $B[P^g] \geqq g > 0$. We consider the case $g=1$. Then for every interior point P of R, $B[P]=2g-2=0$. If there exists a point $P \in R$ at which $B[P^g]>0$, there are single-valued functions $\in \mathfrak{R}$ which have a single pole of order at most g, and such a point P corresponds to the classical Weierstrass point.[7] In the case $g \geqq 2$, we are able to give an example which shows the existence of such a point.

Let F be a two-sheeted Riemann sphere with $2g+2$ branch points $z_1, z_2, \cdots, z_{2g+1}, z_{2g+2}$ which are on the real axis. We remove disjoint n closed intervals which do not contain z_1 on the line $x=\mathrm{Re}\, z_1$ from one sheet of F, and we take the remainder as R, so this is an open Riemann surface of genus g with n boundaries. Then $\frac{1}{z-z_1}$ is a single-valued function $\in \mathfrak{R}$ on R with single pole of order 2 at z_1. Let the point corresponding to z_1 be P, then $A[P^{-2}]>2$, hence $A[P^{-g}]>2$, thus we have $B[P^g]>0$ by Riemann-Roch's theorem.

We do not know yet, in general, whether there be an interior point such that $B[P^g]>0$, and moreover, a number of such points in this case.

In conclusion, the author wishes to express her hearty thanks to Professors A. Kobori and Y. Kusunoki for their kind advices and guidance during this research.

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7) Springer [5, p. 274], and Behnke-Sommer [1, Sätze 39, 40].