62. *Some Properties of Complex Analytic Vector Bundles over Compact Complex Homogeneous Spaces*

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1. This note is a summary of the author's paper which will appear in the Ôsaka Mathematical Journal on the same title. Our concerns are complex analytic vector bundles over C-manifolds in the sense of H. C. Wang [11], and, in particular, homogeneous vector bundles introduced by R. Bott [4]. Mainly by use of Bott's method, we shall investigate some properties of these bundles.

2. Let X be a C-manifold with an almost effective Klein form G/U, where G is a connected complex semi-simple Lie group and U a connected closed complex Lie subgroup of G. Now, let $E=E(\rho, F)$ denote the homogeneous vector bundle defined by a complex analytic representation (ρ, F) of U. Then, the complex vector space $\Gamma_X(E)$ of all sections of E is identified with the set of all holomorphic mappings s of G into F such that

$$s(gu)=\rho(u^{-1})\cdot s(g), \text{ for every } g\in G \text{ and } u\in U.$$

Moreover the induced representation in the sense of Bott, which we denote by $\rho^{\#}$, is defined by

$$(\rho^{\#}(g)s)(g')=s(g^{-1}g')$$

(for every $s\in\Gamma_X(E)$ and $g, g'\in G$) as a complex analytic representation of G over $\Gamma_X(E)$. We define a linear mapping ν of $\Gamma_X(E)$ into F by setting

$$\nu(s)=s(e), \quad (e=\text{the unit element of } G).$$

We say, if ν is surjective, that E has sufficiently many sections. In this case we have an exact sequence as U-modules:

$$(1) \qquad 0\longrightarrow F'\longrightarrow \Gamma_X(E)\stackrel{\nu}{\longrightarrow} F\longrightarrow 0$$

via $\rho^{\#}$ and ρ, as is easily verified, where F' is the kernel of ν. Now assume that $\dim F=m$ and $\dim \Gamma_X(E)=n$, and take the basis $\{\xi_1,\cdots,\xi_n\}$ of $\Gamma_X(E)$ such that $\{\xi_1,\cdots,\xi_{n-m}\}$ span F'. Then, identifying the exact sequence (1) with

$$0\longrightarrow C^{n-m}\longrightarrow C^n\longrightarrow C^m\longrightarrow 0; \quad C^m=C^n/C^{n-m},$$

we can consider $\rho^{\#}$ as a homomorphism of G into $GL(n, C)$ sending U into the subgroup $GL(n, m; C)$ which consists of non-singular matrices leaving C^{n-m} invariant. Thus, we obtain from $\rho^{\#}$, transferring to the coset spaces, a holomorphic mapping f_ρ of X into the complex Grassmann manifold $G(n, m)=GL(n, C)/GL(n, m; C)$. The last manifold

$G(n, m)$ is called the classifying manifold in the so-called classification theorem due to S. Nakano, K. Kodaira and J. P. Serre (cf. [2, 9]), and these authors defined the classifying mapping f_E of X into $G(n, m)$ associated to E in the general (not nec. homogeneous) case. While we have

Theorem 1. *The above defined mapping f_ρ coincides with f_E. Therefore E is induced by f_ρ from the universal quotient bundle W of $G(n, m)$.*

This theorem endows us a unified viewpoint about some known results, which shall be stated in the subsequent sections.

3. Let Θ be the tangent bundle of X, then Θ is the homogeneous vector bundle $E(\mathrm{Ad}\ \mathfrak{g}/\mathfrak{u})$ defined by the linear isotropic representation $(\mathrm{Ad}\ \mathfrak{g}/\mathfrak{u})$ of U. The fact that Θ has sufficiently many sections is equivalent to the homogeneity of the base manifold X. The exact sequence of (1) and the corresponding one of homogeneous vector bundles are, in this case, the following:

$$0 \longrightarrow \mathfrak{u} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{u} \longrightarrow 0;$$
$$0 \longrightarrow L(G) \longrightarrow Q(G) \longrightarrow \Theta \longrightarrow 0,$$

where the latter is Atiyah's exact sequence associated to the coset bundle $G(X, U)$ (cf. [1, 4]). The restriction of the induced representation to $\mathfrak{g} \subset \Gamma_X(\Theta)$ is, as is easily seen, the adjoint representation of G, and the classifying mapping f_Θ associated to Θ is given by $f_\Theta(gU) = \mathrm{Ad}\,(g) \cdot GL(n, m; C)$, where $n = \dim \mathfrak{g}$ and $m = \dim X$. If X is a kählerian in particular, we can consider $\mathfrak{g} = \Gamma_X(\Theta)$ and show that f_Θ is biregular. These situations are nothing but the Gotô's preceding study [5].

4. If E_λ is a homogeneous line bundle over X defined by a character $\lambda \in \mathrm{Hom}\,(U, C^*)$ which has sufficiently many sections, then $G(n, 1)$ is the $(n-1)$-dimensional complex projective space P^{n-1}. Now, let X be kählerian and S the set of simple roots which define the subgroup U (cf. [4, p. 222]) such that the Lie algebra $\mathfrak{u}$ of U is settled by

$$\mathfrak{u} = \mathfrak{v}(S) + \mathfrak{h}(S) + \mathfrak{n}(S),$$

then λ is determined by a weight $\mathring{\lambda} = \sum_{\alpha_i \notin S} p_i \mathring{\Lambda}_i$; $p_i \geqq 0$ and the corresponding mapping f_λ is biregular if and only if all $p_i > 0$, where $\{\mathring{\Lambda}_1, \cdots, \mathring{\Lambda}_l\}$ is the fundamental dominant weights canonically defined by the simple root system $\{\mathring{\alpha}_1, \cdots, \mathring{\alpha}_l\}$. Then the induced representation $\lambda^\#$ of λ is, by a theorem of Bott [4], the irreducible representation of G whose highest weight is $\mathring{\lambda}$. So that the dimension of P^{n-1} comes to be the minimum for the character λ such that $\mathring{\lambda} = \sum_{\alpha_i \notin S} \mathring{\Lambda}_i$ as far as f_λ is biregular. In this case, the classifying mapping f_λ is, by definition, called *the canonical imbedding* of X. For such an imbedding, we have the following theorem:

Theorem 2. *Let X be a kählerian C-manifold whose second Betti number equals to 1, and f_λ the canonical imbedding of X into P^{n-1}. Then, identifying X with its image $f_\lambda(X)$, every positive divisor D of X can be obtained as a hypersurface section of X in P^{n-1}.*

It is readily checked that complex Grassmann manifold and its projective imbedding by using the Plücker coordinates satisfy the assumptions of Theorem 2, so that it is a generalization of a classical theorem of Severi (cf. [7]).

5. Now, there exists one and only one connected closed complex subgroup $\widehat{U}$ of G such that $\widehat{U}$ contains U as a normal subgroup, the factor group $\widehat{U}/U$ is a complex toroidal group and the coset space $\widehat{X}=G/\widehat{U}$ is a kählerian C-manifold. We call the thus obtained principal fibering $X(\widehat{X}, \widehat{U}/U, \phi)$ (ϕ is the natural projection) *the fundamental fibering* of X. We denote by $\mathfrak{O}$ and $\mathfrak{O}^*$ the sheaf of germs of holomorphic functions on X and that of non-vanishing holomorphic functions on X respectively, and similarly by $\widehat{\mathfrak{O}}^*$ the sheaf of germs of non-vanishing holomorpic functions on $\widehat{X}$. Moreover, we have the homomorphism of the abelian group $\mathrm{Hom}(U, C^*)$ of all holomorphic homomorphisms of U into C^* into the group $H^1(X, \mathfrak{O}^*)$ of all complex line bundles over X by attaching the corresponding homogeneous line bundles. Then we have

Theorem 3. *Every complex line bundle over X is homogeneous, and more precisely we have*

(2) $\mathrm{Hom}(U, C^*)\cong H^1(X, \mathfrak{O}^*)$ *under the above homomorphism.*

(3) $H^1(X, \mathfrak{O}^*)=\phi^* H^1(X, \mathfrak{O}^*)+\varepsilon H^1(X, \mathfrak{O})$, *where ε denotes the homomorphism induced from the usual exponential mapping of $\mathfrak{O}$ into $\mathfrak{O}^*$.*

We remark that (2) in this theorem has already obtained by Murakami [9] in a somewhat different method from ours. We add, moreover, that our method combined with the results of Borel-Weil [3] yields Theorem V in Bott [4].

6. Next, we shall discuss the certain reducibility of the structural groups of complex analytic vector bundles over a C-manifold X.

Let $\mathfrak{E}_m(X)$ be the set of (equivalent classes) of all m-dimensional complex analytic vector bundles over X, $\mathfrak{H}_m(X)$ the set of all m-dimensional homogeneous vector bundles over X, $\mathfrak{T}_m(X)$ the subset of $\mathfrak{E}_m(X)$ consisting of vector bundles whose structure group $GL(m, C)$ is reducible to the subgroup $\Delta(m, C)$(=the group of all non-singular triangular matrices), and finally $\mathfrak{S}_m(X)$ the subset of $\mathfrak{E}_m(X)$ which is decomposed into the direct sum of line bundles. Obviously we have

$$\mathfrak{T}_m(X) \supset \mathfrak{S}_m(X) \quad (m \geqq 1),$$

and by Theorem 3

$$\mathfrak{H}_m(X) \supset \mathfrak{S}_m(X) \quad (m \geqq 1).$$

On the other hand, A. Grothendieck [6] has proved that if X is of 1-dimension (i.e. X is a complex projective line P^1), then

$$\mathfrak{E}_m(P^1) = \mathfrak{S}_m(P^1) \quad (m \geqq 1).$$

Therefore we shall investigate the mutual relations between $\mathfrak{E}_m(X)$, $\mathfrak{T}_m(X)$, $\mathfrak{S}_m(X)$ and $\mathfrak{H}_m(X)$ for higher dimensional C-manifolds. We assume that $\dim X > 1$ in the following two propositions.

Proposition 1. *If X is kählerian and U is a maximal solvable subgroup of G (i.e. X is a so-called flag manifold in the generalized sense) then we have*

$$\mathfrak{T}_m(X) \supsetneqq \mathfrak{H}_m(X) \supsetneqq \mathfrak{S}_m(X) \quad (m \geqq 2).$$

Proposition 2. *If X is kählerian and the second Betti number of X equals to 1, then we have*

$$\mathfrak{T}_m(X) = \mathfrak{S}_m(X) \quad (m \geqq 1), \qquad \mathfrak{H}_m(X) \supsetneqq \mathfrak{S}_m(X) \quad (m \geqq \dim X).$$

Combined with the above two propositions, we can deduce the following theorem:

Theorem 4. *A C-manifold X is a complex projective line if and only if the following conditions for vector bundles are satisfied:*

$$\mathfrak{E}_m(X) = \mathfrak{S}_m(X) \quad (m \geqq 1).$$

In [6], Grothendieck stated a conjecture to the effect that the above theorem will be valid for any non-singular projective variety X. Our Theorem 4, therefore, presents a partial answer to his conjecture.

7. Finally we shall discuss the tangential vector bundle Θ of a C-manifold X. For this sake we employ the exact sequence of U (and $\hat{U}$)-modules:

$$0 \longrightarrow \hat{\mathfrak{u}}/\mathfrak{u} \longrightarrow \mathfrak{g}/\mathfrak{u} \longrightarrow \mathfrak{g}/\hat{\mathfrak{u}} \longrightarrow 0,$$

which give rise to the two exact sequences homogeneous vector bundles over X and $\hat{X}$:

$$(4) \qquad 0 \longrightarrow I^r \longrightarrow \Theta \longrightarrow \phi^*\hat{\Theta} \longrightarrow 0 \quad (\text{over } X)$$

$$(5) \qquad 0 \longrightarrow L(X) \longrightarrow Q(X) \longrightarrow \hat{\Theta} \longrightarrow 0 \quad (\text{over } \hat{X}),$$

where I^r is the $r(=\dim \hat{U}/U)$-dimensional trivial bundle, and the latter exact sequence (5) is the Atiyah's exact sequence (cf. [1]) associated to the fundamental fibering. Then, by using (4) and (5), we can prove

Proposition 3. *If we denote by $\mathfrak{a}(X)$ and $\mathfrak{a}(\hat{X})$ the Lie algebras of all holomorphic vector fields on X and $\hat{X}$ respectively, then $\mathfrak{a}(X)$ is the direct sum of $\mathfrak{a}(\hat{X})$ and the (abelian) Lie algebra $\mathfrak{w}$ of $\hat{U}/U$. Therefore the connected analytic automorphism group of X is a com-*

plex reductive Lie group ($\mathfrak{a}(\hat{X})$ *is known to be semi-simple by* [8]).

This improves a result of H. C. Wang [11, Theorem III]. Moreover we have

Proposition 4. *Denoting by* Θ *the sheaf of germs of holomorphic vector fields, we have*

$$\dim H^1(X, \Theta) = r^2 + rs,$$

where $r = \dim X - \dim \hat{X}$ *and* $s = \dim \mathfrak{a}(\hat{X})$.

Note that $H^1(X, \Theta) = \{0\}$ for kählerian C-manifolds X (cf. [4, Theorem VI]), and that this cohomology group has an important meaning in connection with Kodaira-Spencer's deformation theory of complex structures.

8. Finally we shall concern ourselves with the indecomposability of the tangent bundles, and prove the following theorem:

Theorem 5. *The tangential vector bundles of irreducible kählerian C-manifolds are indecomposable.*

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