80. *Some Applications of the Maximum Principle for Subharmonic Functions*

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Let F be hyperbolic Riemann surface and p_0 be a point fixed on F. Let $g(p, p_0)$ be the Green function of F with the pole at p_0 and $h(p, p_0)$ be conjugate to it. G_r is the domain such that $g(p, p_0) > -\log r$ with the boundary C_r. For the points $\tilde{p}, \tilde{p}_0$ on $\tilde{F}$, we define $\tilde{g}(\tilde{p}, \tilde{p}_0)$, $\tilde{h}(\tilde{p}, \tilde{p}_0)$ similarly.

We define the *modulus* of $p, \tilde{p}$ by the relation

$$|p|_F = e^{-g(p,p_0)}, \quad |\tilde{p}|_{\tilde{F}} = e^{-\tilde{g}(p,\tilde{p}_0)},$$

respectively. The ordinary modulus is denoted by '$|\ \ |$'.

1. Let f be an analytic mapping of F into $\tilde{F}$. Then $\tilde{g}(f(p), \tilde{p}_0)$ is harmonic except for the points at which $f(p) = \tilde{p}_0$, and for such points $\tilde{g}(f(p), \tilde{p}_0) = \infty$. Therefore, $\log|f(p)|_{\tilde{F}}$ is subharmonic on F.

Theorem 1 (Schwarz). $|f(p)|_{\tilde{F}} \leqq |p|_F$ *for* $p \in F$.

Proof. Consider the function

$$u(p) = \log|f(p)|_{\tilde{F}} + g(p, p_0).$$

Since $\log|f(p)|_{\tilde{F}}$ is subharmonic and $g(p, p_0)$ is harmonic on $F' = F - p_0$, $u(p)$ is subharmonic on F'. Let z be a local parameter in the neighborhood V of p_0. The function $w(p) = \exp\{-\tilde{g}(f(z), \tilde{p}_0) - i\tilde{h}(f(z), \tilde{p}_0)\}$ is analytic in z. Since we have in V

$$u(z) = -\log|w(z)/z| + u_1(z), \quad u_1 \text{ is harmonic in } V,$$

and $w(0) = 0$, $u(z)$ is subharmonic in V. Thus $u(p)$ is subharmonic on F.

For an arbitrary $r < 1$, $u(p) \leqq \log r$ on C_r. From the maximum principle we obtain the same inequality in G_r. As $r \to 1$, we have $u(p) \leqq 0$ on F, and this proves the theorem.

Corollary 1. *If* $f(p)$ *is an analytic function on* F *such that* $|f(p)| \leqq M$ *and* $f(p_0) = 0$, *then* $|f(p)| \leqq M|p|_F$.

This is easily seen by taking the plane domain $|w| \leqq M$ as $\tilde{F}$ in the theorem.

Theorem 2. *Let* $p_1, p_2, \cdots, p_n$ *be the points such that* $f(p_i) = \tilde{p}_0$, $i = 1, 2, \cdots, n$, *then*

$$|f(p_0)|_{\tilde{F}} \leqq \prod_{i=1}^{n} |p_i|_F.$$

Proof. We assume that $f(p_0) \neq \tilde{p}_0$, otherwise the theorem is trivial. The function $u(p) = \log|f(p)|_{\tilde{F}} + \sum_{i=1}^{n} g(p, p_i)$ is subharmonic on F.

By the argument analogous to the previous proof, we have $u(p)\leqq 0$. Since, for $p=p_0$, $g(p_0, p_i)=-\log|p_i|_F$, we have the theorem.

Corollary 2. *If $f(p)$ is analytic function on F such that $|f|\leqq M$ and $p_1,\cdots,p_n$ are the zeros of f, then*

$$|f(p_0)|\leqq M\prod_{i=1}^{n}|p_i|_F.$$

Theorem 3 (Blaschke). *If $\{p_n\}$ is the sequence such that $f(p_n)=\tilde{p}_0$, then*

$$\sum_{n=1}^{\infty}(1-|p_n|_F)$$

converges.

The proof is parallel to the planer case, and will be omitted.

Theorem 4 (Hadamard). *Let $M(r)=\underset{C_r}{\operatorname{Max}}|f(p)|_{\tilde{F}}$, then $\log M(r)$ is the convex function of $\log r$.*

Proof. Let r_1 and r_2 be such that $0<r_1<r_2<1$. The function

$$u(p)=\frac{\log|f(p)|_{\tilde{F}}-\log M(r_1)}{\log M(r_2)-\log M(r_1)}$$

is subharmonic on F and $\leqq 0$ on C_{r_1}, and $\leqq 1$ on C_{r_2}. Hence, for the harmonic function

$$h(p)=\frac{\log|p|_F-\log r_1}{\log r_2-\log r_1}$$

$u(p)\leqq h(p)$ in $G_{r_2}-G_{r_1}$. This proves the theorem.

2. In the next place, we restrict ourselves to the class $\mathfrak{F}$ of functions f which are analytic on F and whose moduli are single-valued. For simplicity, $|p|_F$ is denoted by r.

Lemma 1. *If $|f|\leqq M$, then*

$$\frac{M(|f(p_0)|-Mr)}{M-r|f(p_0)|}\leqq f(p)\leqq\frac{M(|f(p_0)|+Mr)}{M+r|f(p_0)|}$$

in G_r, $r<1$.

The proof is immediate by Corollary 2.

Theorem 5 (Dieudonné). *Let $|f(p)|\leqq M$ and $f(p_0)=0$. If $d_0=\lim_{r\to 0}|f(p)|/r=1$, then f is univalent in*

$$r<\frac{1}{M+\sqrt{M^2-1}}.$$

Proof. Assume that there exist $p_1, p_2 (p_1\neq p_2,\ r_1\leqq r_2=\rho)$ such that $f(p_1)=f(p_2)=\alpha$. Then the function $G(p)=M^2(\alpha-f(p))/(M^2-\bar{\alpha}f(p))$ belongs to $\mathfrak{F}$, and $|G(p)|\leqq M$, and $G(p_0)=\alpha$, $G(p_1)=G(p_2)=0$.

Hence, from Theorem 2, we have

$$|\alpha|=|G(p_0)|\leqq Mr_1r_2\leqq M\rho^2. \tag{1}$$

Now the function $F(p)=f(p)/w(p)$ belongs to $\mathfrak{F}$ and $|F(p)|\leqq M$ by Theorem 1. Hence, from Lemma 1, we have

$$\frac{M(|F(p_0)|-Mr)}{M-r|F(p_0)|} \leqq \frac{|f(p)|}{r}.$$

Since $F(p_0)=d_0=1$, we have $|f(p)| \geqq Mr(1-Mr)/(M-r)$. At $p=p_2$

$$|\alpha|=|f(p_2)| \geqq \frac{M\rho(1-M\rho)}{M-\rho}. \tag{2}$$

(1), (2) show

$$\rho \geqq \frac{1}{M+\sqrt{M^2-1}}.$$

We shall now generalize Theorem 3.

Theorem 6 (Ostrwski). *Let* $\{p_n\}$ *be zeros of* $f \in \mathfrak{F}$. *The necessary and sufficient condition for* $\sum_{n=1}^{\infty}(1-r_n)$ *to converge is that there exists a positive constant* M *such that for all* r, $r<1$,

$$\int_{C_r} \log|f(re^{i\theta})|\, d\theta \leqq M.$$

Proof. We can assume $f(p_0)=1$. For if $f(p_0)=0$ and its order is n, then setting $F_1(p)=f(p)/(w(p))^n$ and $F(p)=F_1(p)/F_1(p_0)$, we obtain $F \in \mathfrak{F}$ and $F(p_0)=1$.

Consider the function

$$h(p)=\log|f(p)| + \sum_{p_n \in G_r} g_r(p, p_n),$$

where g_r denotes Green function of G_r. Since $h(p)$ is harmonic, $f(p_0)=1$ and $g_r(p, p_n)=g(p, p_n)+\log r$, we have

$$\sum_{r_n<r} \log r/r_n = \frac{1}{2\pi}\int_{C_r} \log|f(re^{i\theta})|\, d\theta.$$

The analogous argument to the planer case concludes the theorem.