77. On the Uniform Approximation by Meromorphic Functions on a Riemann Surface

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The Weierstrass' basic theorem on uniform approximation to continuous function over an interval was first generalized by J. L. Walsh in the complex plane.[1] *Let J be a Jordan curve of the finite z-plane containing in its interior the origin. Then an arbitrary function continuous on J can be uniformly approximated on J by a polynomial in z and $1/z$.*

Let J be a Jordan arc of the finite z-plane. Then an arbitrary function continuous on J can be uniformly approximated on J by a polynomial in z.

A chance to generalize these theorems on an arbitray Riemann surface was given in 1948. H. Behnke and K. Stein discussed the Runge's theorem on a non-compact Riemann surface.[2] By their approach to Runge's theorem we can easily verify the cited theorems on a non-compact Riemann surface.[3]

Let R be an arbitrary non-compact Riemann surface and J be a closed Jordan curve on R. Then an arbitrary function $f(p)$ continuous on J can be uniformly approximated on J by a function meromorphic on R with poles one and at least one in each one of the component of the complement to J. In particular if J does not separate R, $f(p)$ can be approximated uniformly on J by a function holomorphic in R.

In fact, there exists (1) an annular region G of R containing J, (2) an annulus $a\leqq|z|\leqq b$ in the complex plane, (3) a one-one conformal mapping τ of G onto the annulus such that the boundary components of ∂G are mapped on $|z|=a$, $|z|=b$ respectively. Let $J^*=\tau(J)$. Then $f(\tau^{-1}(z))$ is continuous on J^*. Since $f(\tau^{-1}(z))$ can be approximated uniformly on J^* by a polynomial in z and $1/z$, we can find a rational function $R(z)$ such that for every $\varepsilon>0$ $|f(\tau^{-1}(z))-R(z)|<\frac{\varepsilon}{2}$ holds on J^*. Further since $R(\tau(p))$ is holomorphic on $\overline{G}$, we can find a mero-

1) J. L. Walsh: Interpolation and Approximation by Rational Function in the Complex Domain, Amer. Math. Colloq. Pub., 36-48 (1936).

2) H. Behnke and K. Stein: Entwicklungen analytischer Funktionen auf Riemannschen Flächen, Math. Ann., **120**, 430-461 (1948).

3) This is not the case for functions of many variables. Cf. J. Wermer: Polynomial approximation on an arc in C^3, Ann. of Math., **62**, no. 2 (1955).

morphic function $H(P)$ with poles one and at least one in each one of the component of the complement to $\overline{G}$ and it satisfies $|R(\tau(p))-H(p)| < \frac{\varepsilon}{2}$ on $\overline{G}$. We can conclude $|f(p)-H(p)| \leqq |f(p)-R(\tau(p))|+|R(\tau(p))-H(p)|<\varepsilon$. This finishes the proof. Using the above result, we can generalize the Weierstrass' theorem on a Riemann surface.

Let J be a Jordan arc on R. Then an arbitrary function $f(p)$ continuous on J can be uniformly approximated on J by a function holomorphic in R.

To show this, we remark that we can construct a closed Jordan curve J' which contains J as a subarc and separates R. Let the definition of the function $f(p)$ be extended so that $f(p)$ is defined and continuous at every point on J'. The extended function $f(p)$ can be uniformly approximated on J' by a meromorphic function $H(P)$ with poles outside J'. The approximating function $H(P)$ is holomorphic in a region containing J which is simply connected relative to R. So we can find a holomorphic function $g(p)$ in R which approximates $f(p)$ as closely as possible on J.

K. Sakakihara published a paper in which he generalized one of the Walsh's theorems in the following formulation.[4]

Theorem 1. *Let $\overline{\Omega}$ be a compact region bounded by a finite number of mutually non-intersecting Jordan curves $\{\gamma_j\}_{j=1}^n$ on R. Any function continuous on $\overline{\Omega}$ and holomorphic in Ω is uniformly approximated on $\overline{\Omega}$ by a function holomorphic in R if and only if every component of the complement to Ω is non-compact.*

In the following we shall give a proof of the theorem which is of the same idea as the original one, but is a little simpler.

In the first place, we associate with each j an annular region bounded by two closed analytic Jordan curves $\gamma_j^*, \gamma_j^{**}$ as listed before, where γ_j^* is contained in Ω and γ_j^{**} is situated outside $\overline{\Omega}$. Further we take an arbitrary Jordan curve $\gamma_j^\dagger$ which is homotopic to γ_j and contained in the annular region bounded by γ_j^* and γ_j. For the sake of brevity we shall denote the annular region with boundaries $\gamma_j^*, \gamma_j^\dagger$ (or the annular region bounded by $\gamma_j^*, \gamma_j^{**}$) by the notation $(\gamma_j^*, \gamma_j^\dagger)$ (or $(\gamma_j^*, \gamma_j^{**})$). The j-th parametric transformation associated with $(\gamma_j^*, \gamma_j^{**})$ will be denoted by $\tau_j(p)$. Since the function $f(\tau_j^{-1}(z))$ is continuous on the annular region bounded by $\tau_j(\gamma_j^*)$, $\tau_j(\gamma_j)$ and is holomorphic in the interior of the region, we can approximate $f(\tau_j^{-1}(z))$ uniformly by a rational function with poles at $z=0$ and $z=\infty$.[5] So we can assume

4) K. Sakakihara: Meromorphic approximation on Riemann surfaces, Jour. Inst. Polytech. Osaka City Univ., **5**, no. 1 (1954).

5) S. N. Mergelyan: Uniform approximations to function of a complex variable, Uspehi Mathematiceskih Nauk (in Russian), **7**, no. 2 (48), 31–122 (1952).

that $f(p)$ is uniformly approximated on (γ_j^*, γ_j) by a function $g_j(p)$ holomorphic on $(\gamma_j^*, \gamma_j^{**})$, i.e. $|f(p)-g_j(p)|<\varepsilon$ for every p on (γ_j^*, γ_j). Then for every p in $(\gamma_j^*, \gamma_j^\dagger)$

$$\frac{1}{2\pi i}\int_{\gamma_i^\dagger} f(q)d\omega(q,p)=\frac{1}{2\pi i}\int_{\gamma_i^*} f(q)d\omega(q,p)+\delta_{i,j}\cdot f(p)$$

$$\frac{1}{2\pi i}\int_{\gamma_i^{**}} g_i(q)d\omega(q,p)=\frac{1}{2\pi i}\int_{\gamma_i^*} g_i(q)d\omega(q,p)+\delta_{i,j}\cdot g_i(p)$$

where $\delta_{ij}=1$ for $i=j$, $\delta_{ij}=0$ for $i\neq j$ and $d\omega(q,p)$ is a basic differential (Elementardifferential) of R.

Let $g(p)=\sum_{i=1}^{n}\frac{1}{2\pi i}\int_{\gamma_i^{**}} g_i(q)\, d\omega(q,p)$, then $g(p)$ is a holomorphic function on a normal subregion $\bar{\Omega}^{\dagger\dagger}$ containing $\bar{\Omega}$. $\partial\bar{\Omega}^{\dagger\dagger}$ consists of a finite number of analytic Jordan curves each one of which is homotopic to each one of the boundary component of $\partial\Omega$. Put $\max \frac{1}{2\pi}\int_{\gamma_i^*}|d\omega(q,p)| =M_{i,j}$. For every point $p\in(\gamma_j^*, \gamma_j^\dagger)$

$$\left|\frac{1}{2\pi i}\int_{\gamma_i^\dagger} f(q)d\omega(q,p)-\frac{1}{2\pi i}\int_{\gamma_i^{**}} g_i(q)d\omega(q,p)\right|\leqq M_i\cdot\varepsilon$$

where $M_i=\max_j (M_{ij}+1)$. Finally we have

$$f(p)=\frac{1}{2\pi i}\sum_i\int_{\gamma_i^\dagger} f(q)d\omega(q,p)$$

$$\left|f(p)-\frac{1}{2\pi i}\sum_i\int_{\gamma_i^{**}} g_i(q)d\omega(q,p)\right|$$

$$\leqq\left|\frac{1}{2\pi i}\sum_i\int_{\gamma_i^\dagger} f(q)\,d\omega(q,p)-\frac{1}{2\pi i}\sum_i\int_{\gamma_i^{**}} g_i(q)d\omega(q,p)\right|\leqq\sum_i M_i\varepsilon=M\varepsilon.$$

Since $\gamma_j^\dagger$ was chosen arbitrary in (γ_j^*, γ_j), we can prove that $|f(p)-g(p)|<M\varepsilon$ for every point in (γ_j^*, γ_j). The uniform continuity of the function $f(p)$ and $g(p)$ on (γ_j^*, γ_j) assures us the inequality $|f(p)-g(p)|<M'\varepsilon$ on $p\in\bigcup_j\gamma_j$ where M and M' are constants.

By the maximum principle of analytic functions, we have $|f(p)-g(p)|<M'\varepsilon$ on $\bar{\Omega}$. As the region $\bar{\Omega}^{\dagger\dagger}$ bounded by $\{\gamma_j^{\dagger\dagger}\}$ is normal, we can find a function $h(p)$ which is holomorpic in R and satisfies $|g(p)-h(p)|<\varepsilon$ for $p\in\bar{\Omega}$.

From the above estimates, we have

$$|f(p)-h(p)|<(1+M')\varepsilon \quad \text{on } \bar{\Omega}.$$

This proves the sufficiency of the theorem.

To prove the necessity, we need an important lemma due to H.

Behnke and K. Stein.[6]

Lemma. Let R be a non-compact Riemann surface and p_0 be a given point of R and $\rho \geqq 1$ be a given positive integer. Then there exists a one-valued meromorphic function $f(p)$ on R, which is holomorphic except at $p=p_0$, where if z, ζ be the local parameters of p_0, p respectively, then

$$f(p)=\frac{1}{(\zeta-z)^{\rho}}+(\text{holomorphic function at } z).$$

Suppose now there exists a component G of the complement to Ω which is compact relative to R. This component is a region bounded by curves $\{\gamma_j\}_{j=1}^m$ of the components of $\partial\Omega$. In this region, we fix a point p_0 and a meromorphic function $f(p)$ in R, which is holomorphic except at $p=p_0$ and has a pole at $p=p_0$ with residue 1.

In Ω we can describe analytic Jordan curves $\{\gamma_j'\}_{j=1}^m$ such that γ_j' is homotopic to γ_j. The curves $\{\gamma_j'\}_{j=1}^m$ bound a compact region G' such that $G' \supseteqq G$. If there exists a function $g(p)$ which approximates uniformly $f(p)$ on $\overline{\Omega}$. We have

$$\left|\frac{1}{2\pi i}\int_{\partial G'} f(p)\,dz-\frac{1}{2\pi i}\int_{\partial G'} g(p)\,dz\right| \leqq M'''\varepsilon.$$

This concludes $1 \leqq M'''\varepsilon$. This is an impossibility and the necessity is proved by contradiction.

The restriction that the boundary curves are mutually non-intersecting is superfluous. Certainly we have

Theorem 2. *Let $\overline{\Omega}$ be a compact region bounded by a finite number of closed Jordan curves $\{\gamma_j\}_{j=1}^n$ on R, and γ_j, γ_i $(i \neq j)$ may have a single point (but no more) in common. Moreover we suppose that γ_j's belong to the boundaries of the components of the complements to Ω. Any function which is continuous on $\overline{\Omega}$ and holomorphic in Ω is uniformly approximated on $\overline{\Omega}$ by a function holomorphic in R, if and only if every component of the complement to Ω is non-compact.*

A lemma due to R. Osserman[7] states that for every boundary component of Ω there exists a schlichtartig region bounded by a finite number of analytic Jordan curves, and it contains the component. Each one of the boundary components of the region is homotopic to that component of $\partial\Omega$. The procedure of the proof of Theorem 1 can be applied without difficulties.

6) For the proof of this lemma, we refer to the paper of H. Behnke and K. Stein loc. cit. See also M. Tsuji: Behnke-Stein's theorem on the existence of a basic differential on a compact Riemann surface with boundary curves, Comm. Mathematici Univ. Sancti, Pauli, **8**, fasc. 1 (1960).

7) Cf. R. Osserman: A lemma on analytic curves, Pacific Jour. Math., **9**, no. 1 (1959).

By the reasoning of the preceding results, we can prove

Theorem 3. *Let* $\overline{\Omega}$ *be a compact region as stated in Theorem* 1 *and* $\{p_i\}_{i=1}^n$ *be the points one and at least one of which lies in each one of the regions which* $\overline{\Omega}$ *separates* R. *Then every function continuous on* $\overline{\Omega}$ *and holomorphic in* Ω *can be uniformly approximated by a function meromorphic in* R *with poles at* $\{p_i\}_{i=1}^n$.

In the complex plane the general approximation problem on a compact set was completely solved by S. N. Mergelyan. We may conjecture of the validity of the Mergelyan's theorem on a non-compact Riemann surface, but there is not any published proof of it so far as I know.