76. *On the Fundamental Theorem of the Galois Theory for Finite Factors*

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In the previous note [6], it is proved that a Galois theory holds true for finite factors under rather strong conditions. The present note will simplify the proof and clarify the relations of the previous conditions.

1. Following after the terminology of J. Dixmier [1], it is assumed that

(1) *A is a continuous finite factor acting standardly on a separable Hilbert space H.*

According to the representation theory of operator algebras (cf. [1]), (1) implies that H can be seen as the completion of the prehilbert space A^θ equipped with the usual inner product $\langle a^\theta \mid b^\theta \rangle = \tau(ab^*)$ by the standard trace τ, in which 1^θ becomes the trace element of H. Under these circumstances, if G satisfies

(2) *G is an enumerable group of outer automorphisms of A,*

then the following lemma is proved as a sharpening of [6, (1)]:

LEMMA 1. *There exists a unitary representation u_g of G on H such as*

(3) $$x^g = u_g^* x u_g,$$

where x^g means the action of g on $x \in A$.

Although the lemma is proved already by I. E. Segal [7, Theorem 5.3], a sketch of the proof will be given for the sake of convenience. Naturally, the representation u_g is defined by

(4) $$a^\theta u_g = a^{g\theta}.$$

Since

$$\langle a^{g\theta} \mid b^{g\theta} \rangle = \tau(a^g b^{g*}) = \tau(ab^*) = \langle a^\theta \mid b^\theta \rangle,$$

u_g is a unitary operator for each g. Furthermore

$$a^\theta u_g u_h = a^{g\theta} u_h = a^{gh\theta} = a^\theta u_{gh}$$

for all $a \in A$ implies that $g \to u_g$ is a unitary representation of G on H. Finally

$$b^\theta u_g^* a u_g = b^{g^{-1}\theta} a u_g = (ba^g)^\theta = b^\theta a^g$$

implies (3).

LEMMA 2. *Defining by*

(5) $$x'^g = u_g^* x' u_g \quad \text{for} \quad x' \in A,$$

G can be also considered as a group of outer automorphisms on A'.

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The proof of the lemma is essentially the same as [6, Lemma 1]. Since

$$ax'^g = au_g^* x' u_g = u_g^* a^{g^{-1}} x' u_g = u_g^* x' a^{g^{-1}} u_g = u_g^* x' u_g a,$$

for each $a \in A$ and $x' \in A'$, g conserves A', or (5) gives an automorphism on A'. If it is inner, then $u_g^* x' u_g = w'^* x' w'$ for all $x' \in A'$ by a unitary $w' \in A'$, whence $w' u_g^*$ commutes with each $x' \in A'$, or $w' u_g^* \in A$, which is a contradiction since $x^g = w^* x w$ for all $x \in A$ by $w = w' u_g^* \in A$.

LEMMA 3. *The set B of all elements of A which are invariant under G forms a von Neumann subalgebra of A. The commutor B' of B is generated by A' and $\{u_g \mid g \in G\}$.*

The first half of the lemma is obvious. If C' is the von Neumann algebra generated by A' and $\{u_g \mid g \in G\}$, it is clear that C' contains A' and is contained in B'. Hence the commutor C of C' contains B and is contained in A. Therefore, each element c of C belongs to A and commutes with each u_g, whence c belongs to B, that is, $C = B$. This shows $C' = B'$.

LEMMA 4. *If the commutor B' of B in Lemma 3 satisfies*

(6) *B' is a finite factor,*

then u_g is orthogonal to A' in the sense $\tau'(u_g a') = 0$ for each $a' \in A'$ and $g \neq 1$, where τ' is the standard trace of B'.

It is to be noted that (6) implies at once

(7) *B is a factor,*

while the converse implication is not true in general which will be seen by an example of §4. Naturally, it is true that (7) implies (6) if the finiteness of B' is assumed.

To prove the lemma, the conditional expectation ε conditioned by A' in the sense of Umegaki [8] will be employed. By (6), ε projects B' orthogonally onto A'. Hence to prove the lemma it suffices to show that $u_g^\varepsilon = 0$ which follows from (5) by $u_g^\varepsilon a'^g = a' u_g^\varepsilon$ for all $a' \in A'$ using [4, Lemma 1].

The following theorem is a sharpening of [6, Lemmas 3–4]:

THEOREM 1. *If (6) is satisfied by B', then B' is algebraically isomorphic to the crossed product $G \otimes A'$ in the sense of [4].*

If a finite form $\sum_g u_g a_g' = 0$ for $a_g \in A'$, then

$$\tau'[(\textstyle\sum_g u_g a_g')(\sum_g u_g a_g')^*] = 0$$

implies $\tau'(\sum_g a_g' a_g'^*) = 0$ by Lemma 4, whence $a_g' = 0$ for all g. This shows that $\{u_g \mid g \in G\}$ is linearly independent over A' and that the uniqueness of the coefficient a_g of u_g in the expression. This shows also by the natural correspondence $\sum_g u_g a_g' \leftrightarrow \sum_g g \otimes a_g'$, that the finite forms are algebraically isomorphic preserving the traces. Since two finite continuous factors are isomorphic if the metrically dense subalgebras are isomorphic in the trace preserving fashion, the above argument shows that B' is isomorphic to $G \otimes A'$, which proves the theorem.

COROLLARY 1. *Only scalars commute with elements of A' in B', that is, $A \cap B'$ is the scalar multiples of the identity.*

If b is an element of $A \frown B'$ with the expression: $b' = \sum_g u_g a'_g$, then (5) implies $a'^g a'_g = a'_g a'$ for any $a' \in A'$ comparing the coefficients of $b'a'$ and $a'b'$ by virtue of Theorem 1, which implies $a'_g = 0$ for $g \neq 1$ by [4, Lemma 1] and $a_1 = \alpha$ by a certain scalar α since A' is a factor. This proves the corollary.

COROLLARY 2. *Each intermediate von Neumann subalgebra between A' and B' is a subfactor of B', consequently each von Neumann subalgebra between B and A is a subfactor of A.*

If D is the algebra in the hypothesis, then the corollary follows at once by $D \frown D' \leqq D \frown B' \leqq A \frown B'$ and Corollary 1.

THEOREM 2. *If* (1), (2) *and* (6) *are satisfied by A, B and G, then the lattices of all subgroups of G and of all intermediate subfactors B to A are dually isomorphic by the usual Galois correspondence.*

This is now a direct consequence of Theorem 1, Corollary 1 and [4, Theorems 2–3].

2. It seems that the outlook of Theorem 2 generalizes the previous one in [6]. However, the generalization is not substantial which will be seen in the following

THEOREM 3. *If* (1) *and* (2) *are satisfied by A and G, then* (6) *is equivalent to*

(8) *G is finite.*

The proof of the theorem will be divided into two lemmas.

LEMMA 5. *Under the hypothesis of the theorem,* (6) *implies* (8).

Suppose the contrary. Since B' is algebraically isomorphic to $G \otimes A'$ by Theorem 1, there is a subspace M in $G \otimes H$ such that M belongs to the commutor of $G \otimes A'$ and the restriction of $G \otimes A'$ on M is spatially isomorphic to B'. Let n be a natural number greater than $1/\dim M$, where $\dim M$ means the relative dimension of M with respect to $(G \otimes A')'$. Then n-fold copy L of M contains $G \otimes H$. Since B' on H is spatially isomorphic to $G \otimes A'$ on M, the isomorphism carries A' (as a subfactor of B') to a factor on M, whence the n-fold copy $\mathbf{A}'$ of A' acts on L and the restriction of $\mathbf{A}'$ on $G \otimes H$ is possible to identify with $1 \otimes A'$. Consequently, the commutor of $\mathbf{A}'$ on L is isomorphic to $I_n \otimes A$ and is finite, since A' is standard on H by the assumption. On the other hand, by the definition of crossed product $G \otimes A'$, the commutor of $1 \otimes A'$ is isomorphic to $I_\infty \otimes A$, which is infinite. This is impossible since $\mathbf{A}'$ has the finite commutor on L which contains $G \otimes H$.

LEMMA 6. *Under the hypothesis of the theorem,* (8) *implies* (6).

Let m be the order of G. If $\psi = \sum_g (g \otimes 1^g)/\sqrt{m}$, then ψ is a trace element of $1 \otimes A'$ in $G \otimes H$ which follows from the equality

$$<\psi(1\otimes a')\,|\,\psi> = \frac{1}{m}\sum_g <a'^g\,|\,1^g> = \frac{1}{m}\sum_g \tau'(a') = \tau'(a')$$

for $a'\in A'$. If K is the closure of $\psi(1\otimes A')$ in $G\otimes H$, then for $h\in G$

$$\psi(1\otimes a')(h\otimes 1) = \frac{1}{\sqrt{m}}\sum_g (g\otimes 1^g)(h\otimes a'^h)$$
$$= \frac{1}{\sqrt{m}}\sum_g gh\otimes a'^{hg} = \psi(1\otimes a'^h)$$

belongs again K, whence K is invariant under $G\otimes 1$ too. Hence K is invariant under $G\otimes A'$, that is, K belongs to $(G\otimes A')'$. Therefore, the restriction C of $G\otimes A'$ on K is algebraically isomorphic to $(G\otimes A')$ by [1, Prop. 2, p. 19].

The correspondence $a'^g = 1^g\cdot a'\in H \to \psi(1\otimes a')\in K$ for $a'\in A'$ gives a unitary isomorphism u between H and K and clearly it gives

(9) $$u^*a'u = 1\otimes a', \quad u^*u_gu = g\otimes 1.$$

$G\otimes A$ is the continuous finite factor generated by $\{1\otimes a',\ g\otimes 1\}$ on $G\otimes H$, hence C is generated by the same generators on K. On the other hand B' has generators $\{a',\ u_g\}$ by Lemma 3. Thus B' is spatially isomorphic to C by (9) and so necessarily a finite factor.

3. Since every enumerable group, hence every finite group, is representable isomorphically by a group of outer automorphisms of the hyperfinite continuous factor by [5], the Galois correspondence of Theorem 2 between a continuous finite factor and its finite Galois group has effective meaning. Here, it will be shown that a theorem of M. Goldman [2] gives an another example. Goldman proved, among others, for a subfactor B of a continuous finite factor A *if the relative dimension* γ *of* B^θ *with respect to* B' *is* 1/2, *then there is a unitary* $u\in A$ *with* $u^2=1$ *such that*

(i) *each* $a\in A$ *is uniquely expressible in*

(10) $$a = b_1 + ub_2, \quad b_i\in B\ (i=1,2),$$

(ii) *u is orthogonal to B in the sense* $\tau(ub)=0$ *for any* $b\in B$, *and*

(iii) *the automorphism* $x\to x^u$ *conserves B where*

(11) $$x^u = uxu.$$

It will be shown here that A *is the crossed product of B by G* where G is the group of order 2 with the generating element u of (11). If there is a unitary $v\in B$ such as $uxu = vxv^*$ for all $x\in B$, then $x = u^2xu^2 = v^2xv^{*2}$ implies $xv^2 = v^2x$ for all $x\in B$. Since B is a factor, v^2 is a scalar. Hence it is not less general to assume that v^2 is the identity. While $uvu = vvv = v$ implies that v commutes with u, $x = u^2xu^2 = uvxvu$ implies $uvx = xuv$ for all $x\in B$, whence $uv = \lambda$ by some scalar λ, or $u = \lambda v$, which contradicts Goldman's theorem. Hence u of (11) gives an outer automorphism of B. Since the natural correspondence between A and $G\otimes B$ gives an algebraic isomorphism preserving the traces by (i) and (ii), the assertion follows immediately.

Extending the action of u on B' by (11), G becomes also a group of automorphisms of B' which preserves A' in elementwise. It is similar to the above that *the automorphism u is outer.* It is obvious that there are no proper subgroups in G and no proper intermediate subfactors between B and A by [4, Theorem 3], whence it can conclude that *by taking the commutors Goldman's theorem gives an example of the Galois theory for finite factors.*

4. The Galois correspondence does not hold in general for an infinite group of outer automorphisms. We show this fact by an example. In the previous note [5], we have constructed a group of outer automorphisms of the hyperfinite continuous factor which is isomorphic to a previously given enumerably infinite group G. Using the same notations in [5], we shall show the construction in brief. At first we construct a measure space X and a measure preserving ergodic group Γ of transformations acting on X relating to the group G. The same measure space has been already utilized in [3] and Murray and von Neumann have shown that the group G itself is represented as a measure preserving ergodic group of transformations on X [3, pp. 794–796]. As a consequence of this fact, every element $g \in G$ defines an automorphism of the commutative von Neumann algebra $\boldsymbol{M}$ composed of all bounded measurable functions defined on X. On the other hand, Γ is, by the definition, the family of functions $\gamma=[\gamma_g]$ on G satisfying

$$(12) \qquad \gamma_g=\begin{cases} 1 & \text{if } g \text{ belongs to a finite subset } (g_1, g_2, \cdots, g_n) \text{ of } G, \\ 0 & \text{otherwise.} \end{cases}$$

(In the below, γ of (12) will be denoted by $\gamma(g_1, g_2, \cdots, g_n)$.) This Γ becomes a commutative group by the group operation

$$(13) \qquad \gamma+\gamma'=[\gamma_g+\gamma'_g] \ (\text{mod. } 2) \quad \text{for} \quad \gamma=[\gamma_g],\ \gamma'=[\gamma'_g].$$

Then, putting $\gamma^{g_0}(g_1, g_2, \cdots, g_n)=\gamma(g_0g_1, g_0g_2, \cdots, g_0g_n)$, every $g_0 \in G$ defines an automorphism of the group Γ. The actions of $g_0 \in G$ upon $\boldsymbol{M}$ and Γ are extended to an automorphism on the hyperfinite continuous factor $\boldsymbol{A}$ constructed by the Murray-von Neumann method for measure space X and its transformation group Γ. We have shown in [5] that if $g_0 \neq 1$ the automorphism is outer. In the below we shall show that *the invariant elements under every automorphism effected by $g \in G$ are only scalar multiples of the identity.* In fact, since the factor $\boldsymbol{A}$ is nothing but the crossed product $G \otimes \boldsymbol{M}$ (cf. [4, Theorem 4]), every element $a \in \boldsymbol{A}$ is expressed symbolically

$$a=\sum_\gamma \gamma \otimes m_\gamma \qquad (\gamma \in \Gamma,\ m_\gamma \in \boldsymbol{M}).$$

Then the image of a effected by the automorphism g_0 is given by $a^{g_0}=\sum_\gamma \gamma^{g_0} \otimes m^{g_0}$. Hence, if $a^{g_0}=a$,

$$\begin{aligned}\sum_\gamma \gamma^{g_0}(g_1, g_2, \cdots, g_n) \otimes m^{g_0}_{\gamma(g_1,\cdots,g_n)} &= \sum_\gamma \gamma(g_0g_1, \cdots, g_0g_n) \otimes m_{\gamma(g_0g_1,\cdots,g_0g_n)} \\ &= \sum_\gamma \gamma(g_1, \cdots, g_n) \otimes m_{\gamma(g_1,\cdots,g_n)},\end{aligned}$$

i.e. $m_{\gamma(g_0g_1,\cdots,g_0g_n)}=m^{g_0}_{\gamma(g_1,\cdots,g_n)}$. Let us denote by $\|m_\gamma\|_2$ the L_2-norm of m_γ with respect to the measure defined on X. Then

$$\|m_{\gamma(g_1,\cdots,g_n)}\|_2=\|m_{\gamma(g_0g_1,\cdots,g_0g_n)}\|_2.$$

Now we denote the standard trace of $\boldsymbol{A}$ by τ. Then

$$\tau(a^*a)=\sum_\gamma \|m_\gamma\|_2^2<\infty. \tag{14}$$

As G is enumerably infinite, unless a finite set $(g_1, g_2,\cdots, g_n)$ coincides with the empty set ϕ, for infinitely many g, finite sets $\{(gg_1, gg_2,\cdots, gg_n)\}$ are mutually different. Hence to be satisfied the condition (14) by a, $m_{\gamma(gg_1,\cdots,gg_n)}=0$. Thus an invariant element under every automorphism $g\in G$ has the form $\gamma(\phi)\otimes m_{\gamma(\phi)}$. As

$$(\gamma(\phi)\otimes m_{\gamma(\phi)})^g=\gamma(\phi)\otimes m^g_{\gamma(\phi)}=\gamma(\phi)\otimes m_{\gamma(\phi)},$$

$m^g_{\gamma(\phi)}=m_{\gamma(\phi)}$ for every $g\in G$. Because G acts ergodically on X, $m_{\gamma(\phi)}$ must be a constant, that is, the invariant element under every automorphism $g\in G$ is the scalar multiple of the identity.

In the above discussion, $\boldsymbol{A}$ is always the hyperfinite continuous factor and G is arbitrary, whence it is hopeless to suppose that the Galois correspondence holds true for all enumerable infinite groups.

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