74. On the Theory of Non-linear Operators

By Sadayuki YAMAMURO
Hokkaido University

(Comm. by K. KUNUGI, M.J.A., June 13, 1960)

In this note, we consider the eigenvalue problem of some kinds of non-linear integral operators of Hammerstein type. In §1, we will give a general principle, and, in §2, we will apply it to the case of integral operators of Hammerstein type defined on Banach function spaces.

§1. Let R be a Banach space[1] and $\overline{R}$ be its conjugate space. For $\phi \in R$ and $\Phi \in \overline{R}$, we denote the value of Φ at ϕ by (Φ, ϕ).

A functional (in general, non-linear) $F(\phi)$ $(\phi \in R)$ is said to be *Fréchet-differentiable at* ϕ_0 if there exists an operator $\operatorname{grad} F = \mathfrak{f} \in (R \to \overline{R})$ such that

$$F(\phi_0+\phi)-F(\phi_0)=(\mathfrak{f}\phi_0, \phi)+r(\phi_0, \phi),$$

$$\lim_{\|\phi\|\to 0} \frac{|r(\phi_0, \phi)|}{\|\phi\|}=0.$$

$F(\phi)$ is said to be *increasing*[2] (*decreasing*) if

$$\lim_{\|\phi\|\to\infty} F(\phi)=+\infty \quad (-\infty).$$

A linear operator $K \in (R \to \overline{R})$[3] is said to be
symmetric if $(K\phi, \psi)=(K\psi, \phi)$ $(\phi, \psi \in R)$;
positive definite if $(K\phi, \phi) \geqq 0$ $(\phi \in R)$.[4]

Theorem 1. *Let R be a reflexive Banach space and $F(\phi)$ be convex, increasing, $F(0)=0$ and Fréchet-differentiable at any point of R. Let K be completely continuous, symmetric and positive definite. Then, for any number $\rho>0$ there exists a number $\lambda_\rho>0$ and an element $\phi_\rho \neq 0$ such that*

$$K\phi_\rho=\lambda_\rho \mathfrak{f}\phi_\rho, \quad F(\phi_\rho)=\rho.$$

Proof. For any $\rho>0$, put

$$V_\rho=\{\phi \in R: F(\phi) \leqq \rho\}.$$

Then, V_ρ is weakly closed and bounded. In fact, by virtue of continuity and convexity of $F(\phi)$, V_ρ is convex and closed, and the assumption that $F(\phi)$ is increasing implies that V_ρ is bounded.

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1) In this note, we consider only *real* Banach spaces.

2) This definition is due to [2, p. 302].

3) By $(R_1 \to R_2)$, we denote the set of all operators whose domains are R_1 and ranges are in R_2.

4) For these properties of K, see [6], where a kind of eigenvalue problem of such K has been considered.

Moreover, V_ρ contains at least one non-zero element. In fact, take such a $\phi \in R$ that $F(\phi) \neq 0$, then, since $\lim_{\xi \to \infty} F(\xi\phi) = +\infty$, $F(0)=0$ and $F(\xi\phi)$ is continuous as a function of ξ, we can find $\xi > 0$ such that $F(\xi\phi) = \rho$.

Therefore, V_ρ is weakly closed, weakly compact (since R is reflexive) and contains at least two elements.

Next, for the linear operator K, we define the quadratic form $k(\phi)$ by

$$k(\phi) = \frac{1}{2}(K\phi, \phi) \qquad (\phi \in R).$$

Then, $k(\phi)$ is weakly continuous, Fréchet-differentiable at any point of R and

$$\operatorname{grad} k(\phi) = K\phi.$$

Now, consider the functional $k(\phi)$ on the set V_ρ. We can find a $\phi \in V_\rho$ such that $k(\phi) > 0$, because, for a $\phi \in R$ with $k(\phi) > 0$, $\xi\phi \in V_\rho$ for some $\xi > 0$. Therefore, we can find at least one $\phi_\rho \neq 0$ in V_ρ such that $k(\phi_\rho) > 0$ and

$$k(\phi_\rho) \geqq k(\phi) \qquad \text{for every } \phi \in V_\rho.$$

If $F(\phi_\rho) < \rho$, then, by the fact that F is increasing and $F(\xi\phi_\rho)$ is continuous with respect to ξ, there exists such an α that $\alpha > 1$ and $F(\alpha\phi_\rho) = \rho$. Since ϕ_ρ is a maximum point of $k(\phi)$ on V_ρ, we have

$$k(\phi_\rho) \geqq k(\alpha\phi_\rho) = \alpha^2 k(\phi_\rho),$$

which is a contradiction. Therefore, $F(\phi_\rho) = \rho$. Namely, the functional $k(\phi)$ attains its maximum on the boundary of V_ρ. This fact enables us to apply the Lagrange's multiplier rule.[5] Therefore, we can get λ_ρ with

$$K\phi_\rho = \lambda_\rho \mathfrak{f}\phi_\rho.$$

Finally, we will prove that $\lambda_\rho > 0$. At first, we see easily that $\lambda_\rho \neq 0$, because $2k(\phi_\rho) = \lambda_\rho(\mathfrak{f}\phi_\rho, \phi_\rho)$ and $k(\phi_\rho) > 0$. On the other hand, since $F(\phi_\rho) = \rho > 0$, there exists $\varepsilon_0 > 0$ such that $F((1+\varepsilon)\phi_\rho) > 0$ $(0 < \varepsilon < \varepsilon_0)$. Convexity of F implies that

$$F((1+\varepsilon)\phi_\rho) \geqq (1+\varepsilon)F(\phi_\rho) \geqq F(\phi_\rho).$$

Hence it follows that

$$(\mathfrak{f}\phi_\rho, \phi_\rho) = \lim_{\xi \to 0} \frac{F((1+\varepsilon)\phi_\rho) - F(\phi_\rho)}{\varepsilon} \geqq 0.$$

Therefore, $\lambda_\rho > 0$.

§2. Let R be a set of integrable functions defined on the interval $[a, b]$. We assume that R be a Banach space by a norm $\|\phi\|$ $(\phi \in R)$ and the following conditions be satisfied:

R1) *If* $|\phi_1(t)| \leqq |\phi_2(t)|$ *a.e. and* $\phi_2 \in R$, *then* $\phi_1 \in R$ *and* $\|\phi_1\| \leqq \|\phi_2\|$;

R2) *R is reflexive as a Banach space*;

5) For example, see [3, p. 239].

R3) *every Φ of $\overline{R}$ is uniquely represented by an integrable function $\psi(t)$, in other words,*

$$(\Phi, \phi)=\int_a^b \phi(t)\psi(t)dt \qquad (\phi\in R)$$

for an integrable function $\psi(t)$ which is determind uniquely except for a set of measure zero.[6]

In the sequel, we denote Φ by its representation function $\psi(t)$.

Next, we consider a *Carathéodory function* $f(t,u)$ $(a\leqq t\leqq b, -\infty<u<+\infty)$, namely, $f(t,u)$ is measurable with respect to t when u is fixed and is continuous with respect to u when t is fixed.

If the function $f(t,u)$ satisfies the following condition:

f1) *The operator $\mathfrak{f}\phi=f[t,\phi(t)]$ is in $(R\to\overline{R})$ and continuous,* then, we can define a functional $F(\phi)$ as follows:[7]

$$(*) \qquad F(\phi)=\int_a^b\left[\int_0^{\phi(t)} f(t,u)\,du\right]dt.$$

In fact,

$$\begin{aligned}|F(\phi)|&=\left|\int_a^b \phi(t)f[t,\theta(t)\phi(t)]dt\right|, \qquad 0\leqq\theta(t)\leqq 1 \text{ a.e.}\\ &\leqq\|\phi\|\cdot\|\mathfrak{f}(\theta\phi)\|<+\infty\end{aligned}$$

because $\theta\phi\in R$ by R1).

Moreover, $F(\phi)$ is *Fréchet-differentiable at any point in R and* grad $F=\mathfrak{f}$. In fact, for $\phi,\psi\in R$, we have

$$\begin{aligned}&|F(\phi+\psi)-F(\phi)-(\mathfrak{f}\phi,\psi)|\\ &=\left|\int_a^b\left[\int_{\phi(t)}^{\phi(t)+\psi(t)} f(t,u)du\right]dt-\int_a^b\psi(t)f[t,\phi(t)]dt\right|\\ &=\left|\int_a^b\psi(t)\{f[t,\phi(t)+\theta(t)\psi(t)]-f[t,\phi(t)]\}dt\right|\\ &\qquad\qquad\qquad\qquad\qquad\qquad (0\leqq\theta(t)\leqq 1)\\ &\leqq\|\psi\|\cdot\|\mathfrak{f}(\phi+\theta\psi)-\mathfrak{f}\phi\|.\end{aligned}$$

Since $\theta\psi\in R$ and $\lim\limits_{\|\phi\|\to 0}\|\theta\psi\|=0,$

$$\begin{aligned}&\overline{\lim_{\|\phi\|\to 0}}\frac{1}{\|\psi\|}\left|\{F(\phi+\psi)-F(\phi)-(\mathfrak{f}\phi,\psi)\}\right|\\ &\leqq\overline{\lim_{\|\phi\|\to 0}}\|\mathfrak{f}(\phi+\theta\psi)-\mathfrak{f}\phi\|=0,\end{aligned}$$

because, by f1), the operator $\mathfrak{f}$ is continuous.

Now, we can define *the non-linear integral operator of Hammerstein type on $\overline{R}$.*[8] Let a linear operator:

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6) It is well known that the spaces L_p $(p>1)$ and, more generally, Orlicz spaces with (Δ_2)-conditions satisfy these conditions.

7) This procedure is due to [2, p. 81].

8) See [1].

$$K\phi(s)=\int_a^b K(s,t)\phi(t)dt$$

be, as an operator in $(R\rightarrow\bar{R})$, completely continuous, symmetric and positive definite. Let, for a Carathéodory function $g(t,u)$, the operator

$$\mathfrak{g}\psi(t)=g(t,\psi(t))$$

be in $(\bar{R}\rightarrow R)$. Then the operator $H=K\mathfrak{g}$:

$$(**)\qquad H\psi(s)=\int_a^b K(s,t)g(t,\psi(t))dt$$

is called the integral operator of Hammerstein type.

Then, we can state, as an application of Theorem 1, the following

Theorem 2. *Let the integral operator of Hammerstein type* $(**)$ *be defined on* $\bar{R}$ *where the Banach function space* R *satisfies* R1) - R3). *Moreover, we assume that the function* $g(t,u)$ *be strictly increasing with respect to* u *for fixed* t *and the inverse function* $f(t,u)$ *as a function of* u *satisfy* f1) *and the functional* $(*)$ *be increasing. Then, for any* $\rho>0$, *we can find* $\lambda_\rho>0$ *and* $\psi_\rho\neq 0$ *such that*

$$H\psi_\rho=\lambda_\rho\psi_\rho \quad \text{and} \quad F(\mathfrak{g}\psi_\rho)=\rho.$$

Namely, the operator H *has at least one eigenvalue.*

Proof. It is easy to see that the operator $\mathfrak{f}$ generated by the inverse function $f(t,u)$ is the inverse operator of $\mathfrak{g}$. Therefore, for the element ϕ_ρ obtained in Theorem 1, the element $\psi_\rho=\mathfrak{f}\phi_\rho$ is the required eigenfunction.

Remark 1. Naturally, it is possible that the operator H has only one eigenvalue. For example, the operator H considered on L_2 on $[0,1]$ with $K(s,t)=st$ and $g(t,u)=u$ has only one eigenvalue.

Remark 2. If the functional $F(\phi)$ is not increasing, then there exists an operator H which does not have positive eigenvalue. For example, let R be L_3 on $[0,1]$, $g(t,u)=u$ and $K(s,t)=\alpha(s)\alpha(t)$ with $\alpha(t)=1/\sqrt{t}$. Then, the operator H has only one eigenvalue 0.

Remark 3. Necessary and sufficient conditions for the operator $\mathfrak{f}$ to be in $(R\rightarrow\bar{R})$ have been obtained by some authors. For example, when $R=L_p$ $(p>1)$, then $\bar{R}=L_q$, $q=p/p-1$, and $\mathfrak{f}$ is in $(L_p\rightarrow L_q)$ if and only if

$$|f(t,u)|\leqq\phi(t)+\gamma|u|^{\frac{p}{q}} \qquad (-\infty<u<\infty)$$

for some $\gamma>0$ and $\phi(t)\in L_q$.
This result was generalized to the case of Orlicz spaces by Vainberg-Shragin [5], and to the case of conditionally complete vector lattice by Shimogaki [4].

It is easy to see that, in Theorem 1, the same method can be applied to the case when the functional $F(\phi)$ is concave and decreasing. In this case, we can find at least one negative eigenvalue. The eigenvalue problem where the functional $F(\phi)$ is not convex and increasing (or, is not concave and decreasing) will be treated in another paper.

References

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