99. A Note on Subdirect Decompositions of Idempotent Semigroups

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A subsemigroup B of the direct product $B_1 \times B_2 \times \cdots \times B_n$ of bands (i.e. idempotent semigroups) $B_1, B_2, \cdots, B_n$ is called a *subdirect product* of $B_1, B_2, \cdots, B_n$ if every i,

$$\xi_i(B) = B_i$$

where ξ_i is the i-th projection of $B_1 \times B_2 \times \cdots \times B_n$.

Let $\mathfrak{R}_1, \mathfrak{R}_2, \cdots, \mathfrak{R}_m$ be congruences on a band S. Then the set $S^* = \{(\varphi_1(a), \varphi_2(a), \cdots, \varphi_m(a)) : a \in S\}$, where each φ_i is the natural homomorphism of S to $S/\mathfrak{R}_i$, becomes a subdirect product of $S/\mathfrak{R}_1, S/\mathfrak{R}_2, \cdots, S/\mathfrak{R}_m$. Such S^* is called the *natural representation* of S induced by $\mathfrak{R}_1, \mathfrak{R}_2, \cdots, \mathfrak{R}_m$, and denoted by $S/\mathfrak{R}_1 \circ S/\mathfrak{R}_2 \circ \cdots \circ S/\mathfrak{R}_m$. Especially, it has been shown by Birkhoff [1] that if $\mathfrak{R}_1 \cap \mathfrak{R}_2 \cap \cdots \cap \mathfrak{R}_m = 0$,[1] then $S/\mathfrak{R}_1 \circ S/\mathfrak{R}_2 \circ \cdots \circ S/\mathfrak{R}_m$ is an isomorphic representation of S.

Another important type of subdirect product, which is often used in the study of bands, is *spined product* introduced by Kimura [2]:

Let $S_1, S_2, \cdots, S_n$ be bands having Γ as their structure semilattices. And let $\mathfrak{D}_i : S_i \sim \Sigma\{S_i^\gamma : \gamma \in \Gamma\}$, for each i with $1 \leqq i \leqq n$, be the structure decomposition of S_i.[2] Then, the set $S = \bigcup \{S_1^\gamma \times S_2^\gamma \times \cdots \times S_n^\gamma : \gamma \in \Gamma\}$ becomes a subdirect product of $S_1, S_2, \cdots, S_n$. Such S is called the spined product of $S_1, S_2, \cdots, S_n$ with respect to Γ, and denoted by $S_1 \bowtie S_2 \bowtie \cdots \bowtie S_n\ (\Gamma)$.

The main purpose of this paper is to present the following representation theorem which clarifies the relation between such two special kinds of subdirect product.

Theorem. *Let S be a band, and $\mathfrak{D} : S \sim \Sigma\{S_\gamma : \gamma \in \Gamma\}$ its structure decomposition. Let $\mathfrak{R}_1, \mathfrak{R}_2, \cdots, \mathfrak{R}_n$, $n \geqq 2$, be congruences on S.*

If $\mathfrak{R}_1, \mathfrak{R}_2, \cdots, \mathfrak{R}_n$ satisfy

1) The ordering in the set Ω of all congruences on S is as follows: For $\mathfrak{A}, \mathfrak{B} \in \Omega$, $\mathfrak{A} \leqq \mathfrak{B}$ if and only if for $x, y \in S$ $x \mathfrak{A} y$ implies $x \mathfrak{B} y$. The element 0 will denote the least element of Ω in the sense of this ordering.

2) Let S be a band. Then, there exist a semilattice Γ and a disjoint family of rectangular subsemigroups of S indexed by Γ, $\{S_\gamma : \gamma \in \Gamma\}$, such that

$$S = \cup \{S_\gamma : \gamma \in \Gamma\}$$

$$\text{and} \qquad S_\alpha S_\beta \subset S_{\alpha\beta} \quad \text{for } \alpha, \beta \in \Gamma$$

(see McLean [3]). In this case Γ is determined uniquely up to isomorphism, and called the structure semilattice of S. Further this decomposition, say $\mathfrak{D}$, gives a congruence called the structure decomposition of S and denoted by $S \sim \Sigma\{S_\gamma : \gamma \in \Gamma\}$.

$$(\mathrm{C})\quad\begin{cases}(\mathrm{C.1})\ \mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_n \leqq \mathfrak{D},\\ (\mathrm{C.2})\ \mathfrak{N}_1 \cap \mathfrak{N}_2 \cap \cdots \cap \mathfrak{N}_n = 0,\\ (\mathrm{C.3})\ \mathfrak{N}_1 \cap \mathfrak{N}_2 \cap \cdots \cap \mathfrak{N}_i \textit{ and } \mathfrak{N}_{i+1} \textit{ are permutable for all } i,\\ \quad 1 \leqq i \leqq n-1,\\ (\mathrm{C.4})\ (\mathfrak{N}_1 \cap \mathfrak{N}_2 \cap \cdots \cap \mathfrak{N}_i) \cup \mathfrak{N}_{i+1} = \mathfrak{D} \textit{ for all } i,\ 1 \leqq i \leqq n-1,\end{cases}$$

then $S \cong S/\mathfrak{N}_1 \bowtie S/\mathfrak{N}_2 \bowtie \cdots \bowtie S/\mathfrak{N}_n\ (\Gamma)$.[3] *Further, in this case* $S/\mathfrak{N}_1 \circ S/\mathfrak{N}_2 \circ \cdots \circ S/\mathfrak{N}_n = S/\mathfrak{N}_1 \bowtie S/\mathfrak{N}_2 \bowtie \cdots \bowtie S/\mathfrak{N}_n\ (\Gamma)$.

The essential step towards establishing this theorem is the proof of

Lemma. *Let S be a band, and $\mathfrak{D}: S \sim \Sigma\{S_\gamma : \gamma \in \Gamma\}$ its structure decomposition. Let $\mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_n$, $n \geqq 2$, be congruences on S.*

(a) *If $\mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_n$ satisfy* (C. 1), *then for each i with $1 \leqq i \leqq n$ the structure decomposition of $S/\mathfrak{N}_i$ is $S/\mathfrak{N}_i \sim \Sigma\{S_\gamma/\mathfrak{N}_i : \gamma \in \Gamma\}$.*

(b) *If $\mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_n$ satisfy* (C. 1), (C. 3) *and* (C. 4), *then* $S/\mathfrak{N}_1 \circ S/\mathfrak{N}_2 \circ \cdots \circ S/\mathfrak{N}_n = S/\mathfrak{N}_1 \bowtie S/\mathfrak{N}_2 \bowtie \cdots \bowtie S/\mathfrak{N}_n\ (\Gamma)$.

Proof. (a) Let φ_i be the natural homomorphism of S to $S/\mathfrak{N}_i$. Define a relation $\mathfrak{D}_i$ on $S/\mathfrak{N}_i$ as follows: $\varphi_i(x)\mathfrak{D}_i\varphi_i(y)$ if and only if $x'\mathfrak{D}y'$ for some $x' \in \varphi_i(x)$, $y' \in \varphi_i(y)$.

Then, $\mathfrak{D}_i$ gives the structure decomposition of $S/\mathfrak{N}_i$. Denote by $\bar{x}$ the congruence class containing $x \bmod \mathfrak{D}$, and by $\widetilde{\varphi_i(x)}$ the congruence class containing $\varphi_i(x) \bmod \mathfrak{D}_i$.

Then, the mapping ψ_i defined by

$$\psi_i : S/\mathfrak{D} \ni \bar{x} \to \widetilde{\varphi_i(x)} \in S/\mathfrak{N}_i/\mathfrak{D}_i$$

is an isomorphism of $S/\mathfrak{D}$ onto $S/\mathfrak{N}_i/\mathfrak{D}_i$, and $\psi_i(S_\gamma) = S_\gamma/\mathfrak{N}_i$ for all $\gamma \in \Gamma$. Hence, the structure decomposition of $S/\mathfrak{N}_i$ is $S/\mathfrak{N}_i \sim \Sigma\{S_\gamma/\mathfrak{N}_i : \gamma \in \Gamma\}$.

(b) Let $(\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x))$ be an element of $S/\mathfrak{N}_1 \circ S/\mathfrak{N}_2 \circ \cdots \circ S/\mathfrak{N}_n$. Since for each i with $1 \leqq i \leqq n-1$ $\varphi_i(x) \in S_\gamma/\mathfrak{N}_i$ if $x \in S_\gamma$, we have

$$(\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x)) \in S_\gamma/\mathfrak{N}_1 \times S_\gamma/\mathfrak{N}_2 \times \cdots \times S_\gamma/\mathfrak{N}_n \subset S/\mathfrak{N}_1 \bowtie S/\mathfrak{N}_2 \bowtie \cdots \bowtie S/\mathfrak{N}_n\ (\Gamma).$$

Conversely pick up an element $(\varphi_1(a_1), \varphi_2(a_2), \cdots, \varphi_n(a_n))$ from $S/\mathfrak{N}_1 \bowtie S/\mathfrak{N}_2 \bowtie \cdots \bowtie S/\mathfrak{N}_n\ (\Gamma)$. Then, there exists S_γ containing all a_i. Since $\mathfrak{N}_1 \cup \mathfrak{N}_2 = \mathfrak{D}$, we have $a_1(\mathfrak{N}_1 \cup \mathfrak{N}_2)a_2$. Therefore, there exists an element x_2 such that $a_1\mathfrak{N}_1x_2$ and $a_2\mathfrak{N}_2x_2$. Since $(\mathfrak{N}_1 \cap \mathfrak{N}_2) \cup \mathfrak{N}_3 = \mathfrak{D}$ and $\mathfrak{N}_2 \leqq \mathfrak{D}$, we have $x_2((\mathfrak{N}_1 \cap \mathfrak{N}_2) \cup \mathfrak{N}_3)a_3$. Therefore, there exists an element x_3 such that $x_2(\mathfrak{N}_1 \cap \mathfrak{N}_2)x_3$ and $x_3\mathfrak{N}_3a_3$. Hence $a_1\mathfrak{N}_1x_3$, $a_2\mathfrak{N}_2x_3$ and $a_3\mathfrak{N}_3x_3$. Repeating $n-1$ times this process, we obtain an element x_n such that $a_1\mathfrak{N}_1x_n, a_2\mathfrak{N}_2x_n, \cdots, a_n\mathfrak{N}_nx_n$.

Thus $\varphi_i(x_n) = \varphi_i(a_i)$ for all i, and hence

$$\begin{aligned}&(\varphi_1(a_1), \varphi_2(a_2), \cdots, \varphi_n(a_n))\\ &= (\varphi_1(x_n), \varphi_2(x_n), \cdots, \varphi_n(x_n)) \in S/\mathfrak{N}_1 \circ S/\mathfrak{N}_2 \circ \cdots \circ S/\mathfrak{N}_n.\end{aligned}$$

Accordingly, we conclude $S/\mathfrak{N}_1 \circ S/\mathfrak{N}_2 \circ \cdots \circ S/\mathfrak{N}_n = S/\mathfrak{N}_1 \bowtie S/\mathfrak{N}_2 \bowtie \cdots$

3) The notation $\cong$ means the term '$\cdots$ is isomorphic to $\cdots$'.

$\bowtie S/\mathfrak{R}_n\,(\Gamma)$.

Now we can easily prove our theorem by using this lemma and the result of Birkhoff [1].[4] In fact: Since $\mathfrak{R}_1\cap\mathfrak{R}_2\cap\cdots\cap\mathfrak{R}_n=0$, the relation $S\cong S/\mathfrak{R}_1\circ S/\mathfrak{R}_2\circ\cdots\circ S/\mathfrak{R}_n$ follows from the result of Birkhoff [1]. On the other hand, the relation $S/\mathfrak{R}_1\circ S/\mathfrak{R}_2\circ\cdots\circ S/\mathfrak{R}_n=S/\mathfrak{R}_1\bowtie S/\mathfrak{R}_2\bowtie\cdots\bowtie S/\mathfrak{R}_n\,(\Gamma)$ follows from (b) of the lemma. Thus, we have $S\cong S/\mathfrak{R}_1\bowtie S/\mathfrak{R}_2\bowtie\cdots\bowtie S/\mathfrak{R}_n\,(\Gamma)=S/\mathfrak{R}_1\circ S/\mathfrak{R}_2\circ\cdots\circ S/\mathfrak{R}_n$.

Corollary. *Let S be a non-commutative band, and $\mathfrak{D}:S\sim\Sigma\{S_\gamma:\gamma\in\Gamma\}$ its structure decomposition. Let $\mathfrak{R}_1,\mathfrak{R}_2,\cdots,\mathfrak{R}_n$ be congruences on S.*

If $\mathfrak{R}_1,\mathfrak{R}_2,\cdots,\mathfrak{R}_n$ satisfy

(C*)
- (C*. 1) *$\mathfrak{R}_1,\mathfrak{R}_2,\cdots,\mathfrak{R}_n$ are comparable with $\mathfrak{D}$ (i.e. $\mathfrak{R}_i\geqq\mathfrak{D}$ or $\mathfrak{R}_i\leqq\mathfrak{D}$ for each i),*
- (C*. 2) $\mathfrak{R}_1\cap\mathfrak{R}_2\cap\cdots\cap\mathfrak{R}_n=0$,
- (C*. 3) *$\mathfrak{R}_1\cap\mathfrak{R}_2\cap\cdots\cap\mathfrak{R}_i$ and $\mathfrak{R}_{i+1}$ are permutable for all i, $1\leqq i\leqq n-1$,*
- (C*. 4) *$(\mathfrak{R}_1\cap\mathfrak{R}_2\cap\cdots\cap\mathfrak{R}_i)\cup\mathfrak{R}_{i+1}\geqq\mathfrak{D}$ for all i, $1\leqq i\leqq n-1$,*

then $S\cong S/\mathfrak{R}_{i_1}\bowtie S/\mathfrak{R}_{i_2}\bowtie\cdots\bowtie S/\mathfrak{R}_{i_r}\,(\Gamma)=S/\mathfrak{R}_{i_1}\circ S/\mathfrak{R}_{i_2}\circ\cdots\circ S/\mathfrak{R}_{i_r}$ for some $\mathfrak{R}_{i_1},\mathfrak{R}_{i_2},\cdots,\mathfrak{R}_{i_r}$ with $1\leqq i_j\leqq n$.

Application. Let S be a $\Gamma(\varDelta)$-regular band,[5] and $\mathfrak{D}:S\sim\Sigma\{S_\gamma:\gamma\in\Gamma\}$ its structure decomposition. Define relations θ_1,θ_2 on S as follows:

$a\theta_1 b$ if and only if
- $ab=a$ and both a and b are contained in a common S_γ, $\gamma\in\varDelta$,
- or
- $ab=b$ and both a and b are contained in a common S_γ, $\gamma\notin\varDelta$,

$a\theta_2 b$ if and only if
- $ab=a$ and both a and b are contained in a common S_γ, $\gamma\notin\varDelta$,
- or
- $ab=b$ and both a and b are contained in a common S_γ, $\gamma\in\varDelta$.

Then, θ_1,θ_2 are congruences on S which satisfy (C) of the theorem. Hence, $S\cong S/\theta_1\bowtie S/\theta_2(\Gamma)=S/\theta_1\circ S/\theta_2$. This shows that a $\Gamma(\varDelta)$-regular band is isomorphic to the spined product of a $(\Gamma,\Gamma\backslash\varDelta)$-regular band and a $(\Gamma,\varDelta)$-regular band, and especially that a regular band is isomorphic to the spined product of a left regular band and a right regular band.[6]

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4) See p. 92.

5) See Yamada [4].

6) These assertions have been proved also by Yamada [4] and Kimura [2], respectively.

References

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