# 98. *Certain Congruences and the Structure of Some Special Bands*

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(Comm. by K. KUNUGI, M.J.A., July 12, 1960)

1. A *band* is synonymous with an idempotent semigroup. Let $S$ be a band, and $S\sim\Sigma\{S_\gamma:\gamma\in\Gamma\}$ its *structure decomposition* (cf. Kimura [1]). For each subset $\Delta$ of $\Gamma$, we first define the relation $\mathfrak{R}_\Delta$ on $S$ as follows:

$$a\,\mathfrak{R}_\Delta b \text{ if and only if } \begin{cases} ab=a \text{ and both } a \text{ and } b \text{ are contained in} \\ \quad \text{the same } S_\gamma, \gamma\in\Delta, \\ \text{or} \\ ab=b \text{ and both } a \text{ and } b \text{ are contained in} \\ \quad \text{the same } S_\gamma, \gamma\notin\Delta. \end{cases}$$

Then, it is easily seen that $\mathfrak{R}_\Delta$ is an equivalence relation on $S$ but not necessarily a congruence.

The following two theorems have been proved by Kimura [2]:

Theorem I. *$\mathfrak{R}_\phi(\mathfrak{R}_\Gamma)$, where $\phi$ is the empty subset of $\Gamma$, is a congruence on $S$ if and only if $S$ is left (right) semiregular. Further, in this case the quotient semigroup $S/\mathfrak{R}_\phi(S/\mathfrak{R}_\Gamma)$ is left (right) regular.*

Theorem II. *Both $\mathfrak{R}_\phi$ and $\mathfrak{R}_\Gamma$ are congruences on $S$ if and only if $S$ is regular. Further, in this case $S$ is isomorphic to the spined product of $S/\mathfrak{R}_\phi$ and $S/\mathfrak{R}_\Gamma$ with respect to $\Gamma$.*

In this note, we shall present a necessary and sufficient condition for $\mathfrak{R}_\Delta$ to be a congruence on $S$, and make some generalizations of Theorems I and II. However here only the main results and necessary definitions are given, and the proofs are all omitted. We will study them in detail elsewhere.[1)]

Notations and terminologies. If $M$ and $N$ are two sets such that $M\supset N$, then $M\backslash N$ will denote the complement of $N$ in $M$. The notation $\phi$ will denote always the empty set. Throughout the whole paper $S$ will denote a band, unless otherwise mentioned. The structure semilattice of $S$ and the $\gamma$-kernel,[2)] for each $\gamma$ of the structure semilattice, will be denoted by $\Gamma$ and $S_\gamma$ respectively. And the structure decomposition of $S$ will be denoted naturally by $S\sim\Sigma\{S_\gamma:\gamma\in\Gamma\}$. Any other notation or terminology without definition should be referred to [1].

2. Let $\Delta$ be a subset of the structure semilattice $\Gamma$ of $S$, and

1) This is an abstract of the paper which will appear elsewhere.
2) For definition, see [1].

put $\bigcup_{\delta\in\Delta} S_\delta = S(\Delta)$. First of all, we shall define here *$(\Gamma, \Delta)$-semiregularity*, *$\Gamma(\Delta)$-regularity* and *quasi-regularity*.

$S$ is called $(\Gamma, \Delta)$-semiregular or $\Gamma(\Delta)$-regular if it has the following corresponding property (P) or (P*).

$$(\mathrm{P})\quad \begin{cases} cabacba=caba & \text{if } ab\in S(\Delta) \text{ and } abc\in S(\Delta).\\ abac=bac & \text{if } ab\in S(\Delta) \text{ and } abc\notin S(\Delta).\\ caba=cab & \text{if } ab\notin S(\Delta) \text{ and } abc\in S(\Delta).\\ abcabac=abac & \text{if } ab\notin S(\Delta) \text{ and } abc\notin S(\Delta).\end{cases}$$

$$(\mathrm{P}^*)\quad \begin{cases} \left.\begin{matrix} cabacba=caba \\ abcabac=abac \end{matrix}\right\} & \text{if } ab\in S(\Delta) \text{ and } abc\in S(\Delta), \text{ or if } ab\notin S(\Delta) \text{ and } abc\notin S(\Delta).\\ \left.\begin{matrix} caba=cab \\ abac=bac \end{matrix}\right\} & \text{if } ab\in S(\Delta) \text{ and } abc\notin S(\Delta), \text{ or if } ab\notin S(\Delta) \text{ and } abc\in S(\Delta).\end{cases}$$

Further, $S$ is called quasi-regular if it becomes $\Gamma(\Delta)$-regular for some subset $\Delta$ of $\Gamma$.

Of course, it is clear from the definition that for an arbitrary $\Gamma_1\subset\Gamma$, $\Gamma(\Gamma_1)$-regularity is equivalent to $\Gamma(\Gamma\backslash\Gamma_1)$-regularity.

Under these definitions, we have

Lemma 1. *$S$ is $\Gamma(\Delta)$-regular if and only if it is both $(\Gamma, \Delta)$- and $(\Gamma, \Gamma\backslash\Delta)$-semiregular.*

Lemma 2. *$S$ is quasi-regular if and only if it is the class sum of two subsets $A, B$ such that:*

(1) *If $A\ni a$, $axa=a$ and $xax=x$, then $x\in A$.*

(2) *If $B\ni b$, $byb=b$ and $yby=y$, then $y\in B$.*

(3) *If $\left\{\begin{matrix} A\ni ab \text{ and } A\ni abc, \\ or \\ B\ni ab \text{ and } B\ni abc \end{matrix}\right\}$, then $cabacba=caba$ and $abcabac=abac$.*

(4) *If $\left\{\begin{matrix} A\ni ab \text{ and } B\ni abc, \\ or \\ A\ni abc \text{ and } B\ni ab \end{matrix}\right\}$, then $abac=bac$ and $caba=cab$.*

Next, we shall define *bi-regularity* of bands: A band $G$ is called bi-regular if for any given elements $a, b$ of $G$ it satisfies at least one of the relations $aba=ba$ and $aba=ab$.

The global structure of bi-regular bands is given by

Theorem 1. *$S$ is bi-regular if and only if each $\gamma$-kernel is left or right singular.*

Let $G\sim\Sigma\{G_\omega\colon \omega\in\Omega\}$ be the structure decomposition of a bi-regular band $G$. From Theorem 1, every $\omega$-kernel is then left or right singular. Let $\Lambda$ be a subset of $\Omega$.

$G$ is said to be *$(\Omega, \Lambda)$-regular* if it satisfies the following (C):

$$(\mathrm{C})\quad \begin{cases} \text{For } \alpha\in\Lambda,\ G_\alpha \text{ is left singular.}\\ \text{For } \beta\notin\Lambda,\ G_\beta \text{ is right singular.}\end{cases}$$

It is sometimes possible that $G$ is both $(\Omega, \Lambda_1)$- and $(\Omega, \Lambda_2)$-regular for some different subsets $\Lambda_1$ and $\Lambda_2$. Let $G_1$ and $G_2$ be bi-regular bands having the same $\Omega$ as their structure semilattices. Let $G_1 \sim \Sigma\{G_\omega^1: \omega \in \Omega\}$ and $G_2 \sim \Sigma\{G_\omega^2: \omega \in \Omega\}$ be their structure decompositions.

Then, $G_1$ and $G_2$ are called *mutually associated bi-regular bands* if

$$\text{for any given } \omega \in \Omega \begin{cases} G_\omega^1 \text{ is left singular and } G_\omega^2 \text{ is right singular,} \\ \text{or} \\ G_\omega^1 \text{ is right singular and } G_\omega^2 \text{ is left singular.} \end{cases}$$

3. The next two theorems are generalizations of Theorems I and II.

Theorem 2. *$\mathfrak{R}_\Delta$ is a congruence on $S$ if and only if $S$ is $(\Gamma, \Delta)$-semiregular. Further, in this case the quotient semigroup $S/\mathfrak{R}_\Delta$ is a $(\Gamma, \Gamma\backslash\Delta)$-regular band, having $S/\mathfrak{R}_\Delta \sim \Sigma\{S_\gamma/\mathfrak{R}_\Delta: \gamma \in \Gamma\}$ as its structure decomposition.*

Theorem 3. *Both $\mathfrak{R}_\Delta$ and $\mathfrak{R}_{\Gamma\backslash\Delta}$ are congruences on $S$ if and only if $S$ is $\Gamma(\Delta)$-regular. Further, in this case $S$ is isomorphic to the spined product of $S/\mathfrak{R}_\Delta$ and $S/\mathfrak{R}_{\Gamma\backslash\Delta}$ with respect to $\Gamma$.*

Combining Lemmas 1 and 2 with Theorems 2 and 3, we obtain the following corollaries.

Corollary. *If $S$ is $\Gamma(\Delta)$-regular, then it is isomorphic to the spined product of a $(\Gamma, \Gamma\backslash\Delta)$-regular band and a $(\Gamma, \Delta)$-regular band with respect to $\Gamma$.*

Corollary. *If $S$ is quasi-regular, then it is isomorphic to the spined product of mutually associated bi-regular bands with respect to $\Gamma$.*

Corollary. *If $S$ can be decomposed into the class sum of two subsets $A$, $B$ having the properties* (1)–(4) *in Lemma* 2, *then it is isomorphic to the spined product of mutually associated bi-regular bands with respect to $\Gamma$.*

Remark. The existence of a band which is quasi-regular but neither left semiregular nor right semiregular can be verified by giving an example.

## References

[1] N. Kimura: Note on idempotent semigroups. I, Proc. Japan Acad., **33**, 642–645 (1957).
[2] N. Kimura: Ditto. III, Proc. Japan Acad., **34**, 113–114 (1958).