95. Homological Dimension and Product Spaces

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Let X be a topological space and A a closed subset. Let us denote by $H_n(X, A : G)$ the n-dimensional unrestricted Čech homology group of (X, A) with coefficients in an abelian group G. The *homological dimension of X with respect to G* (notation: $D_*(X : G)$) is the largest integer n such that there exists a pair (A, B) of closed subsets of X whose n-dimensional Čech homology group $H_n(A, B : G)$ is not zero. It is obvious that the relation $D_*(X : G) \leqq \dim X$ holds for any space X and any group G, where dim means the covering dimension. A topological space X is called *full-dimensional with respect to an abelian group G* in case $D_*(X : G) = \dim X$. Then the following problem arises naturally:

(*) *Given an abelian group G, what is a space which is full-dimensional with respect to the group?*

The object of this paper is to give an answer to this problem (*) in case X is a locally compact fully normal space and G belongs to a class which includes several important groups. The following theorems hold.

Theorem 1. *Let R be the additive group of all rationals. Then there exists a Cantor manifold M_0 with the property that a locally compact fully normal space X is full-dimensional with respect to R if and only if* $\dim (X \times M_0) = \dim X + \dim M_0$.

Theorem 2. *Let Q_p be the additive group of all rationals reduced* mod 1 *whose denominators are powers of a prime p. Then there exists a Cantor manifold M_p with the property that a locally compact fully normal space X is full-dimensional with respect to Q_p if and only if* $\dim (X \times M_p) = \dim X + \dim M_p$.

A sequence $\mathfrak{a} = \{q_1, q_2, \cdots\}$ is called a *k-sequence* if q_i is a divisor q_{i+1} for each i and $q_i > 1$ for some i. Let us denote by Z_q the cyclic group with order q_i. There exists a natural homomorphism h_i^{i+1} from $Z_{q_{i+1}}$ onto Z_{q_i}, $i = 1, 2, \cdots$. By $Z(\mathfrak{a})$ we mean the limit group of the inverse system $\{Z_{q_i} : h_i^{i+1} | i = 1, 2, \cdots\}$. In a previous paper [2, p. 390], we constructed the Cantor manifold $Q(\mathfrak{a})$ for each k-sequence $\mathfrak{a}$. The following theorem is a consequence of [3, Theorem 1].

Theorem 3. *Let $\mathfrak{a}$ be a sequence. Then a locally compact fully normal space X is full-dimensional with respect to $Z(\mathfrak{a})$ if and only if* $\dim (X \times Q(\mathfrak{a})) = \dim X + \dim Q(\mathfrak{a})$.

Since the cyclic group Z_q with order q is the group $Z(\mathfrak{a})$ for the

k-sequence $\{q, q, \cdots\}$ and $\mathrm{D}_*(X:G)=\mathrm{Max}_\alpha\ \mathrm{D}_*(X:G_\alpha)$ in case G is a direct sum of G_α's, we can characterize a space which is full-dimensional with respect to each finite group. By a consequence of Theorems 1–3, we have the following theorem.

Theorem 4. *Let X and Y be locally compact fully normal spaces and G one of the following groups:* 1) R, 2) Q_p *for each prime* p, 3) $Z(\mathfrak{a})$ *for each k-sequence $\mathfrak{a}$ and* 4) *a direct sum of the groups of* 1)–3). *If* $\mathrm{D}_*(X\times Y:G)=\dim X+\dim Y$, *then X and Y are full-dimensional with respect to G.*

Let us prove only the case $G=Q_p$. By Theorem 2, we have $\dim(X\times Y\times M_p)=\dim X+\dim Y+\dim M_p$. Therefore, both the relations $\dim(X\times M_p)=\dim X+\dim M_p$ and $\dim(Y\times M_p)=\dim Y+\dim M_p$ are true. Thus, X and Y are full-dimensional with respect to Q_p by Theorem 2.

Let R_1 be the additive group of all rationals reduced mod 1. It is well known [4, Theorem 2] that every locally compact fully normal space is full-dimensional with respect to R_1. Since $R_1\approx\sum_p Q_p$, we have the following theorem which is similar to Dyer's theorem [1, Theorem 4.1].*)

Theorem 5. *Let X and Y be locally compact fully normal spaces. If* $\dim(X\times Y)=\dim X+\dim Y$, *then there exists a prime p such that X and Y are full-dimensional with respect to Q_p.*

Let X be a locally compact fully normal space. We shall denote by $\mathrm{D}_{*c}(X:G)$ the homological dimension of X with respect to G defined by making use of Čech homology groups of pairs of *compact subsets* of X with coefficients in G. In general, we do not know whether two dimension functions D_* and D_{*c} are equivalent or not. However, we have the following theorem.

Theorem 6. *Let X be a locally compact fully normal space and G one of the following groups:* 1) R, 2) Q_p *for a prime* p, 3) $Z(\mathfrak{a})$ *for a k-sequence $\mathfrak{a}$ and* 4) *a direct sum of the groups of* 1)–3). *Then Y is full-dimensional with respect to G if and only if* $\mathrm{D}_{*c}(X:G)=\dim X$.

References

[1] E. Dyer: On the dimension of products, Fund. Math., **47**, 141–160 (1959).

[2] Y. Kodama: On a problem of Alexandroff concerning the dimension of product spaces, J. Math. Soc. Japan, **10**, 380–404 (1958).

*) Prof. K. Morita pointed out that our theorem is equivalent to Dyer's as follows. Let R_p be the additive group of all rationals whose denominators are coprime with a prime p. Let R_p^* be the completion, in the p-adic topology, of the ring R_p. Then the following duality holds by general duality theorem [5]: $\mathrm{Hom}_{R_p^*}(H^n(K, L:R_p^*), Q_p)\approx H_n(K, L:Q_p)$ and $\mathrm{Hom}_{R_p^*}(H_n(K, L:Q_p), Q_p)\approx H^n(K, L:R_p^*)$, where H^n means the n-dimensional cohomology group and (K, L) is a pair of finite simplicial complexes. Moreover, the relations $H^n(K, L:R_p^*)\neq 0$ and $H^n(K, L:R_p)\neq 0$ are equivalent.

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