93. *A Theorem on Flat Couples*

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(Comm. by Z. SUETUNA, M.J.A., July 12, 1960)

In this short note, I will prove a theorem in homological algebra and its corollary, which is well known in ideal theory in integral domains.

Throughout this note any ring is assumed to be commutative and have a unit element which acts as the identity operator on any module over the ring. We will call the pair (R, R') of a ring R and its overring R' a flat couple, if R'/R is flat as an R-module [7]. A ring R is called semi-hereditary if every finitely generated ideal of R is R-projective [1]. Then we have the

THEOREM. *Let R be a semi-hereditary ring and R' be an integral (or module finite) extension ring of R. Then, (R, R') is a flat couple.*

The theorem is obtained directly from the following two lemmas.

LEMMA 1. *A semi-hereditary ring is integrally closed in its full ring of quotients.*

PROOF. Let R be a semi-hereditary ring and K be its full ring of quotients. Let x be an element of K and be integral over R and

$$x^n + r_1 x^{n-1} + \cdots + r_n = 0$$

be an equation of integral dependence satisfied by x over R. There exists a non-zerodivisor r of R such that $rx^{n-i} \in R$ for $i = 0, 1, \cdots, n-1$. Since $x^{n+1} = -(r_1 x^n + \cdots + r_n x)$, rx^{n+1} is also in R. Thus we have $rx^i \in R$ for $i = 1, 2, \cdots$. Now, we consider an ideal I of R generated by $(rx^i;\ i = 1, 2, \cdots)$. Since this ideal I is finitely generated (in fact, generated by $rx, rx^2, \cdots, rx^n$) and R is semi-hereditary, I is projective and by Cartan-Eilenberg [1, VII, 3.1] there exist R-homomorphisms $\varphi_i : I \to R$ such that $y = \sum_{i=1}^{n} \varphi_i(y) rx^i$ for all $y \in I$. Thus since $rx \in I$, it follows

$$rx = \sum_{i=1}^{n} \varphi_i(rx) rx^i = \sum_{i=1}^{n} \varphi_i(r^2 x^{i+1}) = \sum_{i=1}^{n} \varphi_i(rx^{i+1}) r,$$

and since r is a non-zerodivisor, we have $x = \sum_{i=1}^{n} \varphi_i(rx^{i+1}) \in R$. This shows that R is integrally closed in K.

Let A be an R-module and a be a non-zero element of A. We say that a is an R-torsion element if $ra = 0$ for some non-zerodivisor r of R, and A is called R-torsion-free if A has no R-torsion element except zero.

LEMMA 2. *A ring R is integrally closed in its full ring of* *quotients, if and only if R'/R is torsion-free as an R-module for each integral (or module finite) extension ring R' of R.*

PROOF. First, we notice that any module finite extension of R is integral over R following M. Nagata ([5] or [6]). Let R' be an arbitrary module finite (or integral) extension of R and further R be integrally closed. Let $\bar{r}'$ $(r' \in R)$ be an element of R'/R and assume that there exists a non-zerodivisor s of R such that $s \cdot \bar{r}' = 0$. Then, sr' is in R, hence r' is an element of the full ring of quotients of R, and moreover r' is integral over R. Thus, since R is integrally closed, r' must be in R, that is $\bar{r}' = 0$ in R'/R. This implies that R'/R is R-torsion-free.

Conversely, let $x = r'/r$ $(r' \neq 0)$ be an element of the full ring of quotients of R and be integral over R. Consider an integral (and module finite) extension $R[x]$ of R and suppose that $R[x]/R$ is R-torsion-free. Then, since $rx \in R$, i.e. $r\bar{x} = 0$ in $R[x]/R$ and r is a non-zerodivisor of R, it follows $\bar{x} = 0$, i.e. $x \in R$. This shows that R is integrally closed.

Now, we return to the theorem. From the above two lemmas, R'/R is R-torsion-free and following a theorem of S. Endo [2], which is a generalization of a theorem of A. Hattori [3], any R-torsion-free module over a semi-hereditary ring is R-flat. Thus, we obtain the result.

If (R, R') is a flat couple, we have $\mathfrak{A}R' \frown R = \mathfrak{A}$ for each ideal $\mathfrak{A}$ of R by Serre [7, Prop. 22]. Thus we have the following corollary and we will prove it to make sure of it.

COROLLARY. *Let R be a semi-hereditary ring, R' be an integral (or module finite) extension of R and $\mathfrak{A}$ be an ideal of R. Then we have*

$$\mathfrak{A}R' \frown R = \mathfrak{A}.$$

PROOF. Let $x = \sum r_i' a_i$ $(a_i \in \mathfrak{A},\ r_i' \in R',\ x \in R)$ be an element of $\mathfrak{A}R' \frown R$. Since R'/R is R-flat, $f: R/\mathfrak{A} (\cong R \underset{R}{\otimes} R/\mathfrak{A}) \to R' \underset{R}{\otimes} R/\mathfrak{A}$ is a monomorphism. Then we have $f(\bar{x}) = 1 \otimes \bar{x} = \sum (1 \otimes \overline{r_i' a_i}) = \sum (r_i' \underset{R}{\otimes} \bar{a}_i) = 0$, this implies $\bar{x} = 0$ in $R/\mathfrak{A}$ i.e. $x \in \mathfrak{A}$. Thus $\mathfrak{A}R' \frown R \subset \mathfrak{A}$ and $\mathfrak{A} \subset \mathfrak{A}R' \frown R$ is obvious.

REMARKS. 1. I express my hearty thanks to Prof. A. Hattori for his following attention. Lemma 2 is not a proposition for "integral", but for "closed". In fact, we can state it more generally as follows. Let R be a ring and K be its full ring of quotients. If an element of an overring of R has a certain property P, which can be considered for an element of an overring of R, we call it a P-element. And if each P-element which is in K is always contained in R, R is called P-closed in K. Then, "if R is P-closed in K, R'/R

is R-torsion-free for any P-extension R' of R", and furthermore if $R[x]$ is a P-extension for any P-element x, the converse holds.

2. In the case of integral domains, a semi-hereditary ring is called a Prüfer ring. It is easily seen that a Prüfer ring is integrally closed, since an integral domain is a valuation ring if and only if it is a local Prüfer ring (cf. [4]). And a torsion-free module over a Prüfer ring is flat by A. Hattori's theorem [3].

References

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