111. On Certain Triangulated Manifolds

By Masahisa ADACHI
Mathematical Institute, Nagoya University
(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1960)

V. Rohlin and A. Schwarz [5] and R. Thom [7] defined the combinatorial Pontrjagin classes of triangulated manifolds and proved the existence of triangulated 8-dimensional manifolds which admit no differentiable structures compatible[1] with their given triangulations. A corresponding result for triangulated 16-dimensional manifolds was proved by K. Srinivasacharyulu [6]. The purpose of this note is to prove the corresponding theorems for the dimensions of the form $4k$ ($2\leqq k\leqq 14$, $k\neq 3$).

In §1 certain triangulated $4k$-dimensional manifolds are constructed and studied. In §2 the theorem is proved.

Our method is quite analogous to that of R. Thom, and closely related with J. Milnor [4]. The word *n-manifold* will always be used for a compact oriented n-dimensional manifold without boundary. The word "differentiable" will be used to mean "differentiable of class C^∞".

1. Let us consider two differentiable mappings of spheres into rotation groups:

$$f_1: S^m\to SO(n+1),\quad f_2: S^n\to SO(m+1).$$

For these mappings Milnor [4] defined the differentiable $(m+n+1)$-manifold $M(f_1,f_2)$ with the following properties:

i) If the mapping f_1 carries S^m into the subgroup $SO(n)\subset SO(n+1)$, then $M(f_1,f_2)$ is a topological sphere.

ii) There exists a differentiable bounded manifold[2] W whose boundary is $M(f_1,f_2)$.

Hereafter we assume that

(*) if $m=n$, the mappings f_1,f_2 both carry S^m into the subgroup $SO(m)\subset SO(m+1)$.

Then $M(f_1,f_2)$ is always a topological $(m+n+1)$-sphere.[3] Furthermore, the differentiable $(m+n+1)$-manifold $M(f_1,f_2)$ has a C^∞-triangulation (L,g), and this C^∞-triangulation can be extended to a C^∞-triangulation (K,f) of the differentiable $(m+n+2)$-manifolds W. Then L is a combinatorial manifold and K is a combinatorial bounded manifold whose boundary is L (cf. Whitehead [8], Milnor [2]).

1) For the precise definition, see Whitehead [8], Milnor [2].

2) bounded manifold=variété à bord.

3) Cf. Milnor [4].

Let T be the space formed from the manifold W by attaching a cone C over the boundary $M(f_1, f_2)$. Since $M(f_1, f_2)$ is a topological $(m+n+1)$-sphere, it follows that T is an $(m+n+1)$-manifold. The triangulation (K, f) of W gives rise to a triangulation (J, h) of the $(m+n+2)$-manifold T. Then we have the following commutative diagram:

$$\begin{array}{ccccc} |L| & \xrightarrow{i_1} & |K| & \xrightarrow{i_2} & |J| \\ g\downarrow & & f\downarrow & & h\downarrow \\ M & \xrightarrow{\bar{i}_1} & W & \xrightarrow{\bar{i}_2} & T \end{array}$$

where $|L|$, $|K|$, $|J|$ are the underlying topological spaces of the simplicial complexes L, K, J, and $i_1, \bar{i}_1, i_2, \bar{i}_2$ are the inclusion maps, and $M = M(f_1, f_2)$.

Hereafter we assume that

$$m = 4r-1, \quad n = 4(k-r)-1,$$
$$1 \leqq r \leqq k-r.$$

We shall study on the triangulated manifold $(J, h; T)$.

a) *Cohomology of* T

The cohomology groups $H^i(T, Z)$ are isomorphic to the cohomology groups $H^i(W, M; Z)$ $(i>0)$. It follows from Milnor [4] that

$$H^i(T, Z) = \begin{cases} Z, & i=0,\ 4r,\ 4(k-r),\ 4k, \\ 0, & \text{otherwise.} \end{cases}$$

We shall denote by α, β the generators in the dimension $4r$, $4(k-r)$, respectively, then $\alpha\beta$ is the generator in the dimension $4k$.

b) *Index of* T

The index $I(T)$ of T is equal to zero. In case $m \neq n$, it is trivial. In case $m=n$, it follows from the assumption $(*)$ (cf. Milnor [4, Lemma 4]).

c) *Combinatorial Pontrjagin classes of* J

Let $i_2 : K \to J$ be the inclusion map. Then the homomorphisms $(i_2)^* : H^q(J, G) \to H^q(K, G)$ induced by i_2 are bijective for $0 \leqq q < 4k$ for any abelian group G. Since L is a triangulated $(4k-1)$-sphere, $j^* : H^q(K, L; G) \to H^q(K, G)$ are bijective for $0 < q < 4k-1$. Moreover we have $j^* : H^q(J, J_0; G) \cong H^q(J, G)$ for $q>0$, where $(J_0, h|J_0)$ is the triangulation of the cone C induced from (J, h). Let Q be the ring of rational numbers. As is remarked in Milnor [3, Chapter XVI, 4], for bounded homology manifold (K, L) we can define the cohomology classes $l_i(K, L) \in H^{4i}(K, L; Q)$ and the combinatorial Pontrjagin classes $p_i(K, L) \in H^{4i}(K, L; Q)$ in the same way as for the homology manifold. We shall denote

$$l_i(K) = j^*(l_i(K, L)),$$
$$p_i(K) = j^*(p_i(K, L)).$$

Let $p_i(J)\in H^{4i}(J, Q)$ be the i-th combinatorial Pontrjagin class of the homology manifold J. Then we have

Lemma 1. For $0<i<k$,

$$(i_2)^*(p_i(J))=p_i(K).$$

Proof. We shall prove the Lemma using the definitions and the notations of Milnor [3, Chapter XVI]. By the definition of the combinatorial Pontrjagin classes, $p_i(J)$ and $p_i(K, L)$ are polynomials of $l_j(J)\in H^{4j}(J, Q)$ and $l_j(K, L)\in H^{4j}(K, L; Q)$, $1\leqq j\leqq i$, respectively. So it is sufficient for us to prove

$$(i_2)^*(l_i(J))=l_i(K)=j^*(l_i(K, L)), \quad \text{for} \quad 0<i<k.$$

Let $\sum^{4k-4i}$ be the boundary of a $(4k-4i+1)$-simplex, and σ be the fundamental cohomology class of $\sum^{4k-4i}$. Let μ, ν be the fundamental homology class of (K, L) and J, respectively. By the definition of $l_i(K, L)$, for any simplicial map $\tilde{\varphi}:(K, L)\to(\sum^{4k-4i}, a)$, where a is a vertex of $\sum^{4k-4i}$, we have

$$<l_i(K, L)\smile(\tilde{\varphi})^*(\sigma),\ \mu>=I(\tilde{\varphi}).$$

Then there exists a simplicial map $\tilde{\psi}$ such that the following diagram is commutative:

$$\begin{array}{ccc} (K, L) & \xrightarrow{\tilde{\varphi}} & (\sum^{4k-4i}, a) \\ & {}_{\tilde{i}_2}\searrow \quad \nearrow_{\tilde{\psi}} & \\ & (J, J_0), & \end{array}$$

where $\tilde{i}_2$ is the inclusion map. Corresponding to this diagram, we have also the following commutative diagram:

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & \sum^{4k-4i} \\ & {}_{i_2}\searrow \quad \nearrow_{\psi} & \\ & J. & \end{array}$$

Then we have

$$\begin{aligned}
&<(j^*)^{-1}\circ(i_2)^*(l_i(J))\smile(\tilde{\varphi})^*(\sigma), \mu> \\
=&<(\tilde{i}_2)^*\circ(j^*)^{-1}(l_i(J))\smile(\tilde{i}_2)^*\circ(\tilde{\psi})^*(\sigma), \mu> \\
=&<(\tilde{i}_2)^*\{(j^*)^{-1}(l_i(J))\smile(\tilde{\psi})^*(\sigma)\}, \mu> \\
=&<(j^*)^{-1}(l_i(J))\smile(\tilde{\psi})^*(\sigma), (\tilde{i}_2)_*(\mu)> \\
=&<(j^*)^{-1}(l_i(J))\smile(j^*)^{-1}\circ\psi^*(\sigma), (\tilde{i}_2)_*(\mu)> \\
=&<(j^*)^{-1}\{l_i(J)\smile\psi^*(\sigma)\}, (\tilde{i}_2)_*(\mu)> \\
=&<l_i(J)\smile\psi^*(\sigma), (j_*)^{-1}\circ(\tilde{i}_2)_*(\mu)> \\
=&<l_i(J)\smile\psi^*(\sigma), \nu> \\
=&I(\psi).
\end{aligned}$$

By the definition of $I(\tilde{\varphi})$, $I(\psi)$, we have $I(\tilde{\varphi})=I(\psi)$. By the uniqueness of $l_i(K, l)$, we obtain the assertion.

2. First recall the index theorem of Hirzebruch [1]. If V is a differentiable $4k$-manifold having Pontrjagin classes $p_1, p_2, \cdots, p_k$, then the index $I(V)$ is equal to $L_k(p_1, p_2, \cdots, p_k)$ $[V]$, where L_k is a certain polynomial. The coefficients s_k of p_k in L_k are expressed in terms of the Bernoulli numbers B_k as follows:

$$s_k = \frac{2^{2k}(2^{2k-1}-1)B_k}{(2k)!}.$$

Let $p_r : \pi_{4r-1}(SO(q)) \to Z$ be the Pontrjagin homomorphisms defined in Milnor [4].

Lemma 2 (Milnor [4]). If $q > 2r$, then there exists an element $(f) \in \pi_{4r-1}(SO(q))$ such that $p_r(f) \neq 0$ and the prime factors of $p_r(f)$ are all less than $2r$.

Combining Lemmas 1, 2, we have

Theorem 1. *Suppose that r is an integer satisfying*

$$k/3 < r \leqq k/2.$$

If the denominator of $s_r s_{k-r}/s_k$ contains a prime factor $\geqq 2(k-r)$, then there exists a triangulated $4k$-manifold T which admits no differentiable structures compatible with its given triangulation (J, h).

Proof. Suppose that the triangulated manifold $(J, h; T)$ admits a differentiable structure $\mathfrak{D}_J$ compatible with the triangulation (J, h). Then $\mathfrak{D}_J$ may define another differentiable structure $\mathfrak{D}_K$ on the underlying manifold of W compatible with the triangulation (K, f). We denote this differentiable manifold by W'. Let

$$\rho^* : H^q(T, Z) \to H^q(T, Q)$$
$$\rho^* : H^q(W, Z) \to H^q(W, Q)$$

be the canonical homomorphisms induced by the injection $\rho : Z \to Q$ of the coefficient groups. Then, by the compatibility of the combinatorial Pontrjagin classes, we have

$$h^* \circ \rho^*(p_i(T)) = p_i(J),$$
$$f^* \circ \rho^*(p_i(W')) = p_i(K) = f^* \circ \rho^*(p_i(W)).$$

However, by Milnor [4] we know

$$p_r(W) = \pm p_r(f_1) \cdot (i_2)^*(\alpha),$$
$$p_{k-r}(W) = \pm p_{k-r}(f_2) \cdot (i_2)^*(\beta).$$

Therefore, by Lemma 1 we have

$$\begin{aligned} h^* \circ \rho^*(p_r(T)) = p_r(J) &= (i_2)^{*-1}(p_r(K)) \\ &= (i_2)^{*-1} \circ f^* \circ \rho^*(p_r(W)) \\ &= h^* \circ (i_2)^{*-1} \circ \rho^*(\pm p_r(f_1) \cdot (i_2)^*(\alpha)) \\ &= \pm p_r(f_1) \cdot h^* \circ \rho^*(\alpha). \end{aligned}$$

Since $H^*(T, Q)$ has no torsion and h^* is bijective, we have

$$p_r(T) = \pm p_r(f_1) \cdot \alpha.$$

Similarly we have

$$p_{k-r}(T) = \pm p_{k-r}(f_2) \cdot \beta.$$

Using the index theorem

$$I(T)=L_k(p_1, p_2,\cdots, p_k)[T],\quad p_i=p_i(T),$$

it follows that[4)]

$$p_k[T]=\pm\frac{s_r s_{k-r}-s_k}{s_k}\cdot p_r p_{k-r}[T],$$

$$p_k[T]=\pm\frac{s_r s_{k-r}-s_k}{s_k}\cdot p_r(f_1)p_{k-r}(f_2),$$

$$p_k[T]=\pm\frac{s_r s_{k-r}}{s_k}\cdot p_r(f_1)p_{k-r}(f_2),\quad \text{mod } 1.$$

By Lemma 2, for $k/3<r$, we can take f_1, f_2 such that $p_r(f_1)p_{k-r}(f_2)\neq 0$ and the prime factors of $p_r(f_1)p_{k-r}(f_2)$ are all less than $2(k-r)$. If $k/3<r\leqq k/2$ and the denominator of $s_r s_{k-r}/s_k$ contains prime factor $\geqq 2(k-r)$, $p_k[T]$ is not an integer. This is a contradiction. Thus we have the theorem.

Theorem 2. *For $2\leqq k\leqq 14$, $k\neq 3$, there exist triangulated $4k$-manifolds $(J, h; T)$ which admit no differentiable structures compatible with their given triangulations (J, h).*

Proof. For such k, it is checked by Milnor [4] that there exists r such that the assumption of Theorem 1 is satisfied.

References

[1] F. Hirzebruch: Neue topologische Methoden in der algebraischen Geometorie, Springer (1956).

[2] J. Milnor: On the relationship between differentiable manifolds and combinatorial manifolds (mimeographed note), Princeton University (1956).

[3] J. Milnor: Lectures on characteristic classes, Princeton University (1958).

[4] J. Milnor: Differentiable structures on spheres, Amer. J. Math., **81**, 962-972 (1959).

[5] V. Rohlin and A. Schwarz: The combinatorial invariance of Pontrjagin classes (in Russian), Doklady Akad. Nauk S.S.S.R., **114**, 490-493 (1957).

[6] K. Srinivasacharyulu: Sur certaines variétés triangulable, C. R. Acad. Sci. Paris, **250**, 2316-2317 (1960).

[7] R. Thom: Les classes caractéristiques de Pontrjagin des variétés triangulées, Symp. Intern. Topologia Algebrica, 54-67 (1958).

[8] J. H. C. Whitehead: On C^1-complexes, Ann. of Math., **41**, 809-824 (1940).

4) The coefficients of $p_r p_{k-r}$ in L_k are calculated in Milnor [4].