109. On the Inequality of Steiner

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We shall be concerned with continuous curves in a Euclidean space $\boldsymbol{R}^m$ of any dimension m. We interpret $\boldsymbol{R}^m$ as a vector space consisting of all the m-tuples $\langle x_1,\cdots,x_m\rangle$ of real numbers, and by a *continuous curve* in $\boldsymbol{R}^m$ we understand a continuous mapping of the real line $\boldsymbol{R}$ into $\boldsymbol{R}^m$ (we may identify $\boldsymbol{R}^1$ with $\boldsymbol{R}$). In what follows, a *curve*, by itself, will always mean a continuous curve in $\boldsymbol{R}^m$ which is locally rectifiable, i.e. rectifiable on every closed interval. This will be tacitly understood throughout. It may be observed that, under this agreement, the sum of any pair of curves is likewise a curve.

We shall call *length-function* for a curve $\varphi(t)$ any real-valued function $F(t)$ defined on $\boldsymbol{R}$ and such that, for every closed interval $I=[a, b]$, the length of the curve φ over I is equal to $F(I)$, where $F(I)$ means as usual the increment $F(b)-F(a)$ of F over I. Thus $F(t)$ is continuous and monotone nondecreasing. Of course, F is not uniquely determined by φ.

Given a pair of curves φ and ψ, let F, G, H be any length-functions for φ, ψ, and $\varphi+\psi$ respectively. Then it is easy to see that, for every closed interval I, we have the relation

$$H(I)\leqq F(I)+G(I). \tag{1}$$

This is the inequality of Steiner. Now it is the object of this paper to *obtain a necessary and sufficient condition for the equality sign to hold in* (1) *for a given interval* I. Although a number of partial or intermediate results in this direction are given in Radó [1], it seems to us that no complete solution of the problem has appeared as yet.

We find it convenient to give a few more definitions. By the *direction* of any nonvanishing vector q of $\boldsymbol{R}^m$ we understand the unit-vector $|q|^{-1}q$. The latter will sometimes be denoted by the symbol $\operatorname{dir} q$. Given a curve φ and a point c of the real line, a unit-vector p of the vector space $\boldsymbol{R}^m$ will be called *tangent direction of* φ *at the point* c, iff (i.e. if and only if) for any positive number ε we can find another positive number δ such that, whenever I is a closed interval containing c and having length less than δ, we have both $\varphi(I)\neq 0$ and $|\operatorname{dir}\varphi(I)-p|<\varepsilon$. The tangent direction of φ at c is obviously uniquely determined when existent, and will be denoted by the symbol $\hat{\varphi}(c)$.

As a notion closely related to that of tangent direction we define further the velocity of a curve φ at a point $c\in\boldsymbol{R}$ as follows. Writing

$\varphi(t)=\langle x_1(t),\cdots, x_m(t)\rangle$, we term φ *derivable at* c iff the coordinate-functions $x_i(t)$ are all derivable at c. When this is the case, we call *velocity of* φ *at* c and denote by $\varphi'(c)$, the vector $\langle x_1'(c),\cdots, x_m'(c)\rangle$. We find at once that if $\varphi'(c)\neq 0$, then the direction of $\varphi'(c)$ is the tangent direction of φ at c.

We say further that a pair of curves φ and ψ are *comparable at a point* $c\in \boldsymbol{R}$, iff there is a positive number K such that we have both $|\varphi(I)|\leqq K|\psi(I)|$ and $|\psi(I)|\leqq K|\varphi(I)|$ for all sufficiently short closed intervals I containing the point c. This is always the case whenever the two curves have nonvanishing velocities at c.

If, finally, $U(t)$ is any real-valued function defined and monotone nondecreasing on $\boldsymbol{R}$, we understand by the symbol U^* the outer Carathéodory measure for $\boldsymbol{R}$, associated with the additive interval-function $U(I)$ in the same manner as expounded on p. 64 of Saks [2].

We are now in a position to enunciate our result in the following form:

Theorem. *Given a pair of curves φ and ψ, let F, G, H be any length-functions for φ, ψ, and $\varphi+\psi$ respectively. In order that the equality*

$$H(I_0)=F(I_0)+G(I_0)$$

should hold for a fixed closed interval $I_0\subset \boldsymbol{R}$, it is necessary and sufficient that we should have

$$F^*(M)=G^*(M)=0 \tag{2}$$

for the set M of the points t of I_0 at each of which the curves φ and ψ are comparable and their respective tangent directions $\hat{\varphi}(t)$ and $\hat{\psi}(t)$ exist without coinciding.

Proof. Let us put $U(t)=F(t)+G(t)+t$ for real numbers t. Then U is a continuous, strictly increasing function which maps $\boldsymbol{R}$ onto $\boldsymbol{R}$, and hence so is its inverse function $\xi(u)$. We find further, for every set X of real numbers,

$$U^*(X)\geqq F^*(X)+G^*(X);$$

this is easily seen by considering the meanings of the appearing quantities. (We could even prove the equality

$$U^*(X)=F^*(X)+G^*(X)+|X|,$$

where $|X|$ denotes the Lebesgue outer measure of X; but this is irrelevant to our present purpose.) On the other hand, the theorem given on p. 100 of Saks [2] implies that $U^*(X)=|U[X]|$ for any $X\subset\boldsymbol{R}$, where $U[X]$ means the image of X under the mapping U. We therefore have

$$F^*(X)=G^*(X)=0 \tag{3}$$

whenever $U[X]$ is a set of measure zero.

With the help of the function $\xi(u)$ introduced above we define now a pair of curves Φ and Ψ by setting, for all points $u\in\boldsymbol{R}$,

$$\Phi(u)=\varphi(\xi(u)) \quad \text{and} \quad \Psi(u)=\psi(\xi(u)).$$

These curves are absolutely continuous, that is, have absolutely continuous coordinate-functions since, for every closed interval J, we have

$$|\Phi(J)|=|\varphi(\xi[J])|\leqq F(\xi[J])\leqq U(\xi[J])=|J|$$

and a similar result for Ψ.

Now the continuous functions P, Q, R defined on the real line by

$$P(u)=F(\xi(u)), \quad Q(u)=G(\xi(u)), \quad R(u)=H(\xi(u))$$

are evidently length-functions for Φ, Ψ, and $\Phi+\Psi$ respectively. As is easily seen further, Φ possesses a tangent direction at a point $u\in\boldsymbol{R}$ iff the curve φ does at the corresponding point $t=\xi(u)$, and when this is the case the two tangent directions coincide; and similarly for the curve Ψ. It is also manifest that Φ and Ψ are comparable at a point u iff φ and ψ are so too at the point $t=\xi(u)$. We find besides, applying once more the theorem quoted above from Saks [2], that, for any set Y of real numbers,

$$P^*(Y)=|P[Y]|=|F(\xi[Y])|=F^*(\xi[Y]). \tag{4}$$

This being so, let us write $I=U[I_0]$, where I_0 is the interval appearing in the theorem, and let us consider the set K of the points of the interval I at which the curves Φ and Ψ, as well as the functions P and Q, are all derivable. Then K is a Borel set on account of a theorem on p. 113 of Saks [2], the coordinate-functions of $\Phi'(u)$ and of $\Psi'(u)$ are B-measurable on K by the same theorem, and the set $I-K$ must be of measure zero in conformity with a well-known theorem of Lebesgue (*vide* p. 115 of Saks [2]).

In view of absolute continuity of Φ and Ψ it follows from Tonelli's theorem on p. 123 of Saks [2] that

$$P(I)=\int_K |\Phi'(u)|\,du, \quad Q(I)=\int_K |\Psi'(u)|\,du, \quad R(I)=\int_K |\Phi'(u)+\Psi'(u)|\,du;$$

whence we derive, writing for simplicity

$$S(u)=|\Phi'(u)|+|\Psi'(u)|-|\Phi'(u)+\Psi'(u)|$$

for $u\in K$ and noting the obvious relations $P(I)=F(I_0)$, $Q(I)=G(I_0)$, and $R(I)=H(I_0)$, that

$$F(I_0)+G(I_0)-H(I_0)=\int_K S(u)du.$$

Now $S(u)$ is always nonnegative on account of the triangular inequality, and thus our task comes to showing that $S(u)$ vanishes almost everywhere in K if and only if the condition (2) holds.

For this purpose, let us classify the general point u of K into three disjoint sets A, B, C according as u fulfils respectively the following three conditions on the velocities $\Phi'(u)$ and $\Psi'(u)$:

(a) $\Phi'(u)=\Psi'(u)=0$; (b) *either* $\Phi'(u)$ *or* $\Psi'(u)$ *vanishes, but not both*; (c) $\Phi'(u)\neq 0$ *and* $\Psi'(u)\neq 0$.

These are all Borel sets together with K, and $S(u)$ vanishes everywhere in A and B. For brevity, we shall write K_0, A_0, B_0, C_0 for the sets $\xi[K]$, $\xi[A]$, $\xi[B]$, $\xi[C]$. Since $|I-K|=0$, we then infer from (3) that

$$(5) \qquad F^*(I_0-K_0)=0.$$

Now the decomposition theorem of de la Vallée Poussin on p. 127 of Saks [2] gives, for any Borel set $Y\subset K$,

$$P^*(Y)=\int_Y P'(u)\,du.$$

But Tonelli's theorm (*loc. cit.*) informs us that $P'(u)=|\Phi'(u)|$ almost everywhere in K. Consequently we find in virtue of (4) that, for any Borel set $Y\subset K$,

$$(6) \qquad F^*(\xi[Y])=\int_Y |\Phi'(u)|\,du.$$

Taking account of (a) we readily deduce hereby that

$$(7) \qquad F^*(A_0)=0.$$

On the other hand the condition (b) implies that the curves Φ and Ψ cannot be comparable anywhere in B, or what amounts to the same thing by what has already been observed, that the set B_0 contains no points of comparability of the curves φ and ψ. Consequently the set M defined in the theorem is disjoint from B_0. In view of (5) and (7) it follows now at once that

$$\begin{aligned} F^*(M) &\leqq F^*(M-K_0)+F^*(MA_0)+F^*(MB_0)+F^*(MC_0) \\ &\leqq F^*(I_0-K_0)+F^*(A_0)+F^*(MB_0)+F^*(MC_0)=F^*(MC_0). \end{aligned}$$

Combining this with $F^*(M)\geqq F^*(MC_0)$ we get $F^*(M)=F^*(MC_0)$, and its counterpart $G^*(M)=G^*(MC_0)$ must also be true by symmetry. The condition (2) is thus equivalent to

$$(8) \qquad F^*(MC_0)=G^*(MC_0)=0;$$

and so the proof will be complete if we show that (8) is equivalent to $|N|=0$, where we denote by N the set of the points u of K at which $S(u)>0$. Thus defined, N lies in C as observed in the above, and is a Borel set since $S(u)$ is a B-measurable function on K.

For any point u of C the relation $S(u)=0$ means, in view of the condition (c), that there is a positive real number λ fulfilling $\Psi'(u)=\lambda\Phi'(u)$. In other words, a point u of C belongs to the set N when and only when $\operatorname{dir}\Phi'(u)\neq\operatorname{dir}\Psi'(u)$. On the other hand (c) clearly implies that at all points u of C the curves Φ and Ψ are comparable and have $\operatorname{dir}\Phi'(u)$ and $\operatorname{dir}\Psi'(u)$ for their respective tangent directions. Accordingly the curves φ and ψ are comparable everywhere in C_0

and their tangent directions at any point $t \in C_0$ are given respectively by dir $\Phi'(u)$ and dir $\Psi'(u)$, where we write $u = U(t)$. It follows at once that $MC_0 = \xi[N]$. This together with (3) reveals that the relation $|N| = 0$ implies (8).

The proof draws now to a close, it only remaining to be shown that (8) implies $|N| = 0$. On account of (6) we deduce from (8) that

$$\int_N |\Phi'(u)| \, du = 0.$$

But since $\Phi'(u) \neq 0$ for all $u \in N$ in accordance with (c), this cannot take place unless the set N is of measure zero.

References

[1] Radó, Tibor,: Length and Area, Amer. Math. Soc. Colloquim Publications, **30** (1948).

[2] Saks, Stanisław,: Theory of the Integral (second revised edition, 1937), reprinted by Hafner Publishing Company.