136. Note on Metrizability and n-Dimensionality

By Keiô NAGAMI

Ehime University, Matsuyama

(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1960)

This note gives firstly some metrizability conditions which are trivial corollaries of J. Nagata's general metrization theorem (§ 1). Our metrization theorem can be applied to criterions for n-dimensionality of metric spaces with the aid of the concept 'cushioned refinements' obtained by E. Michael (§ 2). One of the benefits of this interesting concept is to include both closed closure-preserving refinements and open star-refinements (cf. Remark 1.5). Thus our criterion for n-dimensionality provides us with more general form than [4, Theorem 7.2 and Theorem 7.5] where closed closure-preserving refinements and open star-refinements are essentially used respectively. It is to be noted that throughout this note a covering need not be open.

§ 1. **Metrizability.** **Lemma 1.1** (J. Nagata's general metrization theorem [5, Theorem 1]). *In order that a topological space R be metrizable it is necessary and sufficient that one can assign a neighborhood basis $\{U_i(x);\ i=1,2,\cdots\}$, neighborhood systems $\{S_i^1(x);\ i=1,2,\cdots\}$ and $\{S_i^2(x);\ i=1,2,\cdots\}$ satisfying the following conditions.*

(1) *$y\notin U_i(x)$ implies $S_i^2(y)\frown S_i^1(x)=\phi$(=the empty-set).*

(2) *$y\in S_i^1(x)$ implies $S_i^2(y)\subset U_i(x)$.*

Theorem 1.2. *In order that a topological space R be metrizable it is necessary and sufficient that there exists a sequence of coverings $\mathfrak{H}_i$, $i=1,2,\cdots$, of R which satisfies the following conditions.*

(3) *For any point x of R and any neighborhood U of x there exists i with $\mathrm{S}(x,\mathfrak{H}_i)$[1] $\subset U$.*

(4) *For any point x of R and any i there exists j with $x\notin \overline{\mathrm{S}(R-\mathrm{S}(x,\mathfrak{H}_i),\mathfrak{H}_j)}$.*

Proof. Since the necessity is clear, we prove only the sufficiency.

i) When $x\notin\overline{\mathrm{S}(R-\mathrm{S}(x,\mathfrak{H}_i),\mathfrak{H}_j)}$, let us put

$$U_{ij}(x)=\mathrm{S}(x,\mathfrak{H}_i),$$
$$S_{ij}^1(x)=R-\mathrm{S}(R-\mathrm{S}(x,\mathfrak{H}_i),\mathfrak{H}_j),$$
$$S_{ij}^2(x)=\mathrm{S}(x,\mathfrak{H}_j).$$

ii) When $x\in\overline{\mathrm{S}(R-\mathrm{S}(x,\mathfrak{H}_i),\mathfrak{H}_j)}$, let us put

$$U_{ij}(x)=S_{ij}^1(x)=R,$$
$$S_{ij}^2(x)=\mathrm{S}(x,\mathfrak{H}_i).$$

1) $\mathrm{S}(x,\mathfrak{H}_i)=\smile\{H;\ x\in H\in\mathfrak{H}_i\}$. When R_1 is a subset of R, $\mathrm{S}(R_1,\mathfrak{H}_i)=\smile\{H;\ R_1\frown H\neq\phi,\ H\in\mathfrak{H}_i\}$.

Then by (3) and (4) $\{U_{ij}(x);\ i, j=1, 2, \cdots\}$ is a neighborhood basis of x and $\{S_{ij}^k(x);\ i, j=1, 2, \cdots\}$, $k=1, 2$, are neighborhood systems of x. To verify (1), let $y \notin U_{ij}(x)$. Then $y \notin S(x, \mathfrak{H}_i)$ and hence $y \in R - S(x, \mathfrak{H}_i)$. Since $S_{ij}^1(x) \frown S_{ij}^2(y) = (R - S(R - S(x, \mathfrak{H}_i), \mathfrak{H}_j)) \frown S(y, \mathfrak{H}_j) \subset (R - S(R - S(x, \mathfrak{H}_i), \mathfrak{H}_j)) \frown S(R - S(x, \mathfrak{H}_i), \mathfrak{H}_j) = \phi$, (1) holds good. Next let us verify (2). In case ii), (2) clearly holds. Hence we consider only the case i). Let $y \in S_{ij}^1(x) = R - S(R - S(x, \mathfrak{H}_i), \mathfrak{H}_j)$. Then $y \notin S(R - S(x, \mathfrak{H}_i), \mathfrak{H}_j)$ and hence $S(y, \mathfrak{H}_j) \frown (R - S(x, \mathfrak{H}_i)) = \phi$. Therefore $S_{ij}^2(y) = S(x, \mathfrak{H}_j) \subset S(x, \mathfrak{H}_i) = U_{ij}(x)$ and (2) holds good. Thus by Nagata's general metrization theorem we conclude that R is metrizable and the proof is completed.

Remark 1.3. It is almost evident that the condition (4) of the theorem can be replaced with any one of the following conditions.

(5) For any point x of R, $\{S(S(x, \mathfrak{H}_i), \mathfrak{H}_j);\ i, j=1, 2, \cdots\}$ forms a neighborhood basis of x.

(6) For any point x and any i there exists j and a neighborhood U of x such that $S(D, \mathfrak{H}_j) \subset S(x, \mathfrak{H}_i)$.

Remark 1.4. When $\mathfrak{H}_i$, $i=1, 2, \cdots$, are open coverings, Theorem 1.2 has already been proved by K. Morita [2, Theorem 4]. I am informed by Professor Morita that his theorem yields Nagata's general metrization theorem by very simple argument.

Definition 1.5 (E. Michael [1]). Let $\mathfrak{H}_1 = \{H_\alpha;\ \alpha \in A\}$ and $\mathfrak{H}_2 = \{H_\beta;\ \beta \in B\}$ be coverings of a topological space R. If a map f of B into A satisfies the condition:

(7) for any $\beta \in B$ $H_\beta \subset H_{f(\beta)}$,

then we call f a refine-map. If $f: B \to A$ satisfies the condition:

(8) for any subset B_1 of B $\overline{\smile\{H_\beta;\ \beta \in B_1\}} \subset \smile\{H_\alpha;\ \alpha \in f(B_1)\}$,

then we call f a *cushion-map*. When there is a cushion-map, we call $\mathfrak{H}_2$ a *cushioned refinement* of $\mathfrak{H}_1$.

Remark 1.6 (E. Michael [1]). The following two cases are remarkable examples of cushioned refinements. i) When $\mathfrak{H}_2$ is open and $\mathfrak{H}_2^* = \{S(H, \mathfrak{H}_2);\ H \in \mathfrak{H}_2\}$ refines $\mathfrak{H}_1$, $\mathfrak{H}_2$ is a cushioned refinement of $\mathfrak{H}_1$. ii) When $\overline{\mathfrak{H}}_2 = \{\overline{H};\ H \in \mathfrak{H}_2\}$ refines $\mathfrak{H}_1$ and $\mathfrak{H}_2$ is a *closure-preserving* covering, i.e. one in which $\smile\{\overline{H}_\beta;\ \beta \in B_1\} = \overline{\smile\{H_\beta;\ \beta \in B_1\}}$ holds for any subset B_1 of B, then $\mathfrak{H}_2$ is a cushioned refinement of $\mathfrak{H}_1$. Especially when $\overline{\mathfrak{H}}_2$ refines $\mathfrak{H}_1$ and $\mathfrak{H}_2$ is locally finite, then $\mathfrak{H}_2$ is a cushioned refinement of $\mathfrak{H}_1$.

Lemma 1.7. *Let $\mathfrak{H}_1 = \{H_\alpha;\ \alpha \in A\}$ and $\mathfrak{H}_2 = \{H_\beta;\ \beta \in B\}$ be coverings of a topological space R such that $\mathfrak{H}_2$ is a cushioned refinement of $\mathfrak{H}_1$. Then the following mutually equivalent conditions evidently hold.*

(9) *For any point x of R, $x \notin \overline{S(R - S(x, \mathfrak{H}_1), \mathfrak{H}_2)}$.*

(10) *For any point x of R there exists a neighborhood D of x with $S(D, \mathfrak{H}_2) \subset S(x, \mathfrak{H}_1)$.*

By this lemma we get at once the following proposition as a corollary of Theorem 1.2.

Corollary 1.8. *In order that a topological space R be metrizable it is necessary and sufficient that there exists a sequence of coverings $\mathfrak{H}_i$, $i=1, 2, \cdots$, of R which satisfies the following conditions.*

(11) *For any point x of R and any neighborhood U of x there exists i with* $S(x, \mathfrak{H}_i) \subset U$.

(12) *For any i $\mathfrak{H}_{i+1}$ is a cushioned refinement of $\mathfrak{H}_i$.*

Let us take this occasion to apply Nagata's general metrization theorem to criterions for the normality of open coverings. To this aim we restate Lemma 1.1 as follows.

Lemma 1.9. *In order that an open covering $\mathfrak{H}_0$ of a topological space R be normal*[2] *it is necessary and sufficient that there exist for any point x of R neighborhood systems $\{U_i(x);\ i=1, 2, \cdots\}$, $\{S_i^k(x);\ i=1, 2, \cdots\}$ $(k=1, 2)$ of x which satisfy the following conditions.*

(13) *For any $H \in \mathfrak{H}_0$ and any $x \in H$ there exists i with $U_i(x) \subset H$.*

(14) *For any i, j and any $x \in R$ there exists k with $U_k(x) \subset S_i^1(x) \frown S_j^2(x)$.*

(15) *For any i, any $x \in R$ and any $y \in S_i^1(x)$ it holds that $S_i^2(y) \subset U_i(x)$.*

(16) *For any i, any $x \in R$ and any $y \notin U_i(x)$ it holds that $S_i^2(y) \frown S_i^1(x) = \phi$.*

By an analogous way to the proof of Theorem 1.2 we get the following two propositions as corollaries of the above lemma.

Theorem 1.10. *An open covering $\mathfrak{H}_0$ of a topological space R is normal if there exists a sequence of coverings $\mathfrak{H}_i$, $i=1, 2, \cdots$, which satisfies the following conditions.*

(17) *$\mathfrak{H}_i$ is a cushioned refinement of $\mathfrak{H}_{i-1}$, $i=1, 2, \cdots$.*

(18) *$x, y \in S(z, \mathfrak{H}_{i+1})$ yields $x \in S(y, \mathfrak{H}_i)$, $i=1, 2, \cdots$.*

(19) *For any i, any subcollection $\mathfrak{F}_i$ of $\mathfrak{H}_i$ and any $x \in R - \overline{\smile\{H;\ H \in \mathfrak{F}_i\}}$ there exists j such that $S(x, \mathfrak{H}_j) \subset R - \overline{\smile\{H;\ H \in \mathfrak{F}_i\}}$.*

(20) *For any $H \in \mathfrak{H}_0$ and any $x \in H$ there exists j with $S(x, \mathfrak{H}_j) \subset H$.*

Theorem 1.11. *An open covering $\mathfrak{H}_0$ of a topological space R is normal if there exists a sequence of coverings $\mathfrak{H}_i$, $i=1, 2, \cdots$, which satisfies the following conditions.*

(21) *For any $H \in \mathfrak{H}_0$ and any $x \in H$ there exists j with $S(x, \mathfrak{H}_j) \subset H$.*

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2) An open covering $\mathfrak{H}_0$ is called *normal* if there exists a sequence of open coverings $\mathfrak{H}_i$, $i=1, 2, \cdots$, which satisfies the following mutually equivalent conditions. i) For any i $\mathfrak{H}_{i+1}^* = \{S(H, \mathfrak{H}_{i+1});\ H \in \mathfrak{H}_{i+1}\}$ refines $\mathfrak{H}_i$. ii) For any i $\mathfrak{H}_i$ is locally finite and elementary open and $\mathfrak{H}_{i+1}^*$ refines $\mathfrak{H}_i$, where $\mathfrak{H}_i$ is called elementary open if every element $H \in \mathfrak{H}_i$ is expressible as $H=\{x;\ f(x)>0\}$ for suitable real valued continuous function f defined on R. iii) There exists a pseudo-metric ρ on R such that $\{S_1(x)=\{y;\ \rho(x, y)<1\};\ x \in R\}$ refines $\mathfrak{H}_0$.

(22) *For any i and any $x \in R$* $\mathrm{S}(x, \mathfrak{H}_i)$ *is a neighborhood of x.*

(23) *For any i and any $x \in R$ there exist j and k with* $\mathrm{S}(\mathrm{S}(x, \mathfrak{H}_j), \mathfrak{H}_k) \subset \mathrm{S}(x, \mathfrak{H}_i)$.

§2. **n-Dimensionality.** **Theorem 2.1.** *In order that a topological space $R \neq \phi$ be a metrizable space with* $\dim R \leq n$[3] *it is necessary and sufficient that there exists a sequence of coverings $\mathfrak{H}_i$, $i=1, 2, \cdots$, of R which satisfies the following conditions.*

(24) *For any i $\mathfrak{H}_{i+1}$ is a cushioned refinement of $\mathfrak{H}_i$.*

(25) $\liminf_i \mathrm{order}\,(x, \mathfrak{H}_i)$[4] $\leq n+1$ *for every $x \in R$.*

(26) *For any $x \in R$ and any neighborhood U of x there exists i with* $\mathrm{S}(x, \mathfrak{H}_i) \subset U$.

Proof. a) Since the necessity is clear, we prove only the sufficiency. Let $\mathfrak{H}_i = \{H(\alpha); \alpha \in A_i\}$, $i=1, 2, \cdots$, be a sequence of coverings satisfying (24), (25), (26). By Corollary 1.8 R is metrizable.[5] Let us prove $\dim R \leq n$.

b) Let $f_{i+1,i}: A_{i+1} \to A_i$ $(i=1, 2, \cdots)$ be a cushion-map. For any i, j with $i > j$ put $f_{ij} = f_{j+1,j} \cdots f_{i-1,i-2} f_{i,i-1}$. Let $f_{ii}: A_i \to A_i$ be the identity-map. Consider A_i $(i=1, 2, \cdots)$ as a topological space with the discrete topology. Then $\{A_i, f_{ij}\} = \{A_i, f_{ij}; i, j=1, 2, \cdots, i > j\}$ forms an inverse limiting system. Let $A = \{a = (\alpha_1, \alpha_2, \cdots); a \in \lim \{A_i, f_{ij}\}, \bigcap_{i=1}^{\infty} H(\alpha_i) \neq \phi\}$ and π_i $(i=1, 2, \cdots)$ the projection of A into A_i.

c) Let $\varphi: A \to R$ be a map defined by $\varphi(a) = \bigcap_{i=1}^{\infty} H(\pi_i(a))$. Then $\varphi(a)$ consists of one and only one point for every $a \in A$ by the definition of A and the condition (26). To prove φ is onto, let x be an arbitrary point of R. Let $A_i(x) = \{\alpha; x \in H(\alpha) \in \mathfrak{H}_i\}$. Then $A_i(x) \neq \phi$ for every i and $f_{ij}(A_i(x)) \subset A_j(x)$ for any i, j with $i > j$. Hence $\{A_i(x), f_{ij} | A_i(x)\}$ forms an inverse limiting system. By the condition (25) there exists a sequence of increasing positive integers $1', 2', \cdots$ such that $A_{i'}(x)$ consists of a finite number of indices for every i. Hence $\lim \{A_{i'}(x), f_{i'j'} | A_{i'}(x)\} \neq \phi$. Since $A(x) = \lim \{A_i(x), f_{ij} | A_i(x)\}$ is essentially the same with $\lim \{A_{i'}(x), f_{i'j'} | A_{i'}(x)\}$, we get $A(x) \neq \phi$. Let a be an arbitrary point of $A(x)$. Then evidently $\varphi(a) = x$ and we know that φ is onto.

d) Let i be an arbitrary positive integer and α an arbitrary index of $\pi_i(A)$. Put

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3) $\dim R$ denotes the covering dimension of R. We call $\dim R \leq n$ if and only if every finite open covering of R can be refined by an open covering whose order is at most $n+1$. As to the order of a covering see the footnote below.

4) $\mathrm{order}\,(x, \mathfrak{H}_i)$ is the number of elements of $\mathfrak{H}_i$ which contain x. The order of $\mathfrak{H}_i$ is the supremum of $\{\mathrm{order}\,(x, \mathfrak{H}_i); x \in R\}$.

5) If we hope, the metrizability of R can be proved once more at the end of the proof with no use of the results of §1. Hence it is not absolutely necessary to utilize Corollary 1.8.

(27) $F(\alpha)=\bigcap_{j=i}^{\infty}(\cup\{H(\beta); f_{ji}(\beta)=\alpha\})$.

Since for every $j\geq i$ $\overline{\cup\{H(\beta); f_{j+1,i}(\beta)=\alpha\}}\subset\cup\{H(\gamma); f_{ji}(\gamma)=\alpha\}$ by the condition (24),[6] $F(\alpha)\subset\bigcap_{j=i}^{\infty}\overline{(\cup\{H(\beta); f_{ji}(\beta)=\alpha\})}\subset\bigcap_{j=i+1}^{\infty}\overline{(\cup\{H(\beta); f_{ji}(\beta)=\alpha\})}$ $\subset\bigcap_{j=i}^{\infty}(\cup\{H(\beta); f_{ji}(\beta)=\alpha\})=F(\alpha)$ and hence $F(\alpha)$ is closed.

Let x be an arbitrary point of R; then by c) there exists a point a of A with $\varphi(a)=x$. Evidently $x\in F(\pi_i(a))$. Thus

(28) $\mathfrak{F}_i=\{F(\alpha); \alpha\in\pi_i(A)\}$, $i=1,2,\cdots$

is a sequence of closed coverings of R. Moreover for any j, i with $j>i$ and any $\alpha\in\pi_j(A)$ it evidently holds that $F(\alpha)\subset F(f_{ji}(\alpha))$. Thus $\mathfrak{F}_j$ refines $\mathfrak{F}_i$.

e) Let us show that for any i the order of $\mathfrak{F}_i$ is at most $n+1$. Let x be an arbitrary point of R. By the condition (25) there exists j with $j\geq i$ such that order$(x,\mathfrak{H}_j)\leq n+1$. Since for any $\alpha\in\pi_j(A)$ $F(\alpha)\subset H(\alpha)$, we get order$(x, F_j)\leq n+1$. Since order$(x,\mathfrak{F}_i)\leq$order$(x,\mathfrak{F}_j)$, we get order$(x,\mathfrak{F}_i)\leq n+1$.

f) To show that for any i $\mathfrak{F}_i$ is closure-preserving, let B be an arbitrary subset of $\pi_i(A)$. We put

(29) $E_{j\alpha}=\cup\{H(\beta); \beta\in f_{ji}^{-1}(\alpha)\}$, $j=i, i+1,\cdots, \alpha\in\pi_i(A)$,

(30) $\Delta=\cup\{F(\alpha); \alpha\in B\}=\bigcup_{\alpha\in B}(\bigcap_{j=i}^{\infty}E_{j\alpha})$,

(31) $\nabla=\bigcap_{j=i}^{\infty}(\bigcup_{\alpha\in B}E_{j\alpha})$.

Then evidently $\Delta\subset\nabla$.

To prove $\nabla\subset\Delta$ let x be an arbitrary point of ∇. By the condition (25) there exists $k\geq i$ with order$(x,\mathfrak{H}_k)\leq n+1$. Since for every $\alpha\in\pi_i(A)$ $E_{i\alpha}\supset E_{i+1,\alpha}\supset\cdots$, we get for any $l\geq k$ order$(x,\{E_{l\alpha}; \alpha\in B\})$ $\leq$order$(x,\{E_{k\alpha}; \alpha\in B\})\leq$order$(x,\mathfrak{H}_k)\leq n+1$. Let

(32) $B_j(x)=\{\alpha; x\in E_{j\alpha}, \alpha\in B\}$, $j=i, i+1,\cdots$.

Since $B_i(x)\supset B_{i+1}(x)\supset\cdots$ and for every $l\geq k$ $B_l(x)$ is a non-empty finite set of indices, $\bigcap_{j=i}^{\infty}B_j(x)$ is non-empty. Hence there exists an index $\alpha(x)$ of B such that $\alpha(x)\in B_j(x)$ for every $j\geq i$. Therefore $x\in\bigcap_{j=i}^{\infty}E_{j,\alpha(x)}$ and hence $x\in\bigcup_{\alpha\in B}(\bigcap_{j=i}^{\infty}E_{j\alpha})=\Delta$. Thus we get $\nabla\subset\Delta$ and hence

(33) $$\Delta=\nabla.$$

Since $\nabla=\bigcap_{j=i}^{\infty}(\bigcup_{\alpha\in B}(\cup\{H(\beta);\ \beta\in f_{ji}^{-1}(\alpha)\}))=\bigcap_{j=i}^{\infty}(\cup\{H(\beta);\ \beta\in f_{ji}^{-1}(B)\})$ $\subset\bigcap_{j=i+1}^{\infty}\overline{(\cup\{H(\beta); \beta\in f_{ji}^{-1}(B)\})}\subset\nabla$, we get $\bar{\nabla}=\nabla$. By (33) we also get $\bar{\Delta}=\Delta$. Thus we can conclude that $\mathfrak{F}_i$ is closure-preserving.

g) For every i $\mathfrak{F}_i$ is a closure-preserving point-finite closed

6) Evidently for any j, i with $j<i$ $\mathfrak{H}_j$ is a cushioned refinement of $\mathfrak{H}_i$ with a cushion-map f_{ji}.

covering by d), e), f). Hence it is almost evident that $\mathfrak{F}_i$ is a locally finite closed covering. Moreover the sequence $\mathfrak{F}_i$, $i=1,2,\cdots$, satisfies the following three conditions.

(34) For any i $\mathfrak{F}_{i+1}$ refines $\mathfrak{F}_i$.

(35) For any i order $(x,\mathfrak{F}_i)\leq n+1$ for every $x\in R$.

(36) For any $x\in R$ and any neighborhood U of x there exists i with $S(x,\mathfrak{F}_i)\subset U$.

Therefore we get $\dim R\leq n$ by [4, Theorem 3.2][7] and the proof is completed.

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7) It is also proved that the sequence $\mathfrak{F}_i$, $i=1,2,\cdots$, constructed here satisfies Morita's n-dimensionality condition [3, Theorem 2].