134. Some Topological Properties on Royden's Compactification of a Riemann Surface

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1. Let R be an open Riemann surface and $M(R)$ be the totality of bounded a.c.T. functions on R with finite Dirichlet integrals and $M_0(R)$ be the totality of functions in $M(R)$ with compact supports. We denote by $M_\Delta(R)$ the closure of $M_0(R)$ in BD-convergence topology, where a sequence $\{\varphi_\nu\}$ converges to φ in BD-convergence topology if the sequence $\{\varphi_\nu\}$ is bounded and converges to φ uniformly on each compact subset of R and the sequence $\left\{\iint_R d(\varphi_\nu-\varphi)\wedge \overline{{}^*d(\varphi_\nu-\varphi)}\right\}$ converges to zero.

Royden's compactification R^* of R is the unique compact Hausdorff space containing R as its open and dense topological subspace such that any function in $M(R)$ can be uniquely extended to R^* so as to be continuous on R^*. The (*Royden's*) *ideal boundary* of R is defined by R^*-R and denoted by ∂R. The compact set $\Delta=\{p\in R^*; f(p)=0$ for all f in $M_\Delta(R)\}$ is a part of ∂R and called the *harmonic boundary* of R. We also say that $\partial R-\Delta$ is the non-harmonic boundary of R. These notions are introduced by Royden [3]. Our formulation above mentioned is different from that in [3] but equivalent to that of Royden. Details are in [1].

In this note we state some topological properties of R^* and solve a question raised in [3].

2. Consider a normal exhaustion $\{R_n\}_1^\infty$ of R in the sense of Pfluger [2]. The open set $R-\overline{R}_n$ is decomposed into a finite number of non-compact connected components $K_1^{(n)}, K_2^{(n)},\cdots,K_{N(n)}^{(n)}$. A determining sequence is a sequence $\{K_{i_n}^{(n)}\}_1^\infty$ such that

$$K_{i_1}^{(1)}\supset K_{i_2}^{(2)}\supset\cdots\supset K_{i_n}^{(n)}\supset K_{i_{n+1}}^{(n+1)}\supset\cdots \quad . \tag{1}$$

If we fix an exhaustion $\{R_n\}_1^\infty$, then the totality of determining sequences corresponds in a one-to-one and onto manner to the totality of *ends* of R in the sense of Kerékjártó-Stoïlow [2]. Let $\{E_k\}$ be the decomposition of ∂R into connected components. First we show

Theorem 1. *The decomposition $\{E_k\}$ can be regarded as the totality of ends of R in the sense of Kerékjártó-Stoïlow.*

Proof. An end is determined by a sequence (1). Then the intersection $\alpha=\bigcap_1^\infty\overline{K}_{i_n}^{(n)}$ is a continuum in ∂R, since each $\overline{K}_{i_n}^{(n)}$ is a connected compact set in R^*. There exists a component E_k in $\{E_k\}$ such that $E_k \frown \alpha$ is not empty. By the definition of E_k, α is contained in E_k. Now we show that $\alpha = E_k$. Contrary to the assertion, assume the existence of a point p_0 in $E_k - \alpha$. Then we can find a positive integer n such that p_0 lies outside $\overline{K}_{i_n}^{(n)}$. Take a closed Jordan curve C in $K_{i_n}^{(n)}$ which is homologous to $\Gamma = R \frown (\overline{K}_{i_n}^{(n)} - K_{i_n}^{(n)})$ in R. We denote by D the compact domain in R which is surrounded by Γ and C. Let $w(p)$ be the harmonic function defined in D with boundary value 1 on Γ and 0 on C. We put

$$f(p) = \begin{cases} 1 & \text{on } R - K_{i_n}^{(n)}; \\ w(p) & \text{on } D; \\ 0 & \text{on } K_{i_n}^{(n)} - D. \end{cases}$$

Clearly $f(p)$ is contained in $M(R)$ and hence continuous on R^*. Thus $f(p) = 0$ on $\partial R \frown \overline{K}_{i_n}^{(n)}$. In particular, f is continuous on E_k and takes only two values 0 and 1. Obviously $f = 0$ on $\alpha \subset E_k$ and 1 at $p_0 \in E_k$. This is absurd, since E_k is connected. Hence we have proved that for any end whose determining sequence is $\{K_{i_n}^{(n)}\}$, we can find an E_k in $\{E_k\}$ such that

$$E_k = \bigcap_1^\infty \overline{K}_{i_n}^{(n)}. \tag{2}$$

Conversely we assert that for any E_k in $\{E_k\}$, there exists an end whose determining sequence is $\{K_{i_n}^{(n)}\}$ satisfying (2). For the aim, take an E_k in $\{E_k\}$ and a point p_0 in E_k. Using the function $f(p)$ above defined, we conclude that the sets $\overline{K}_1^{(n)}, \overline{K}_2^{(n)}, \cdots, \overline{K}_{N(n)}^{(n)}$ are mutually disjoint. Hence the set $K_{i_n}^{(n)}$ containing p_0 in its closure is uniquely determined for each n. Therefore, the sequence $\{K_{i_n}^{(n)}\}$ such that $p_0 \in \overline{K}_{i_n}^{(n)}$ is uniquely determined and is a determining sequence. By a similar argument as above, we see that $E_k = \bigcap_1^\infty \overline{K}_{i_n}^{(n)}$. Hence we have proved that for any E_k in $\{E_k\}$ there exists an end whose determining sequence is (1) and satisfies (2).

Thus $\{E_k\}$ corresponds to the totality of ends of R in a one-to-one and onto manner. Q.E.D.

From this theorem, we may call E_k an *end* of R. Incidently we have also seen in the above proof that any dividing cycle of R divides ∂R.

3. Let $\mathfrak{U}^{K_{i_n}^{(n)}}$ be the totality of non-negative superharmonic functions u on $K_{i_n}^{(n)}$ such that at any point p_0 in $(\overline{K}_{i_n}^{(n)} - K_{i_n}^{(n)}) \frown R$

$$\varliminf_{p \to p_0} u(p) \geq 1.$$

We set

$$w(p; K_{i_n}^{(n)}) = \inf (u(p);\ u \in \mathfrak{U}^{K_{i_n}^{(n)}})$$

on $K_{i_n}^{(n)}$. By Perrons' theorem, we see that $w(p; K_{i_n}^{(n)})$ is a harmonic function on $K_{i_n}^{(n)}$ with boundary value 1 on $(\overline{K}_{i_n}^{(n)} - K_{i_n}^{(n)}) \frown R$. If none of

$w(p; K_{i_n}^{(n)})$ is constant for a sequence (1), then we say that the end determined by (1), i.e. $E_k=\bigcap_1^\infty \overline{K}_{i_n}^{(n)}$ is *hyperbolic*. If this is not the case, we say that it is *parabolic*. Here we remark that the Green's function $g(p, p_0)$ of R is continuous on R^* except a point p_0 in R and vanishes on Δ. Now we prove

Theorem 2. *The following three conditions are mutually equivalent:*

(a) *E_k is a hyperbolic end;*

(b) *$E_k \frown \Delta$ is non-empty;*

(c) $\inf (g(p, p_0); p\in E_k)=0$.

Proof. (a) implies (b). Contrary to the assertion, assume that $E_k\frown\Delta$ is empty. Then $\overline{K}_{i_n}^{(n)}\frown\Delta$ is empty for all sufficiently large n. For each point p in $\overline{K}_{i_n}^{(n)}$, there exists a function f_p in $M_\Delta(R)$ such that $f_p(p)\neq 0$. Considering f_p^2 instead of f_p, we may assume $f_p\geq 0$ on R^*. Since $\overline{K}_{i_n}^{(n)}$ is compact, we can find a finite number of points p_ν in $\overline{K}_{i_n}^{(n)}$ and a positive number c such that

$$g=\sum_\nu f_{p_\nu}\geq c>0$$

on $K_{i_n}^{(n)}$ and clearly g is in $M_\Delta(R)$. Let $\widehat{K}_{i_n}^{(n)}$ be Schottky's double of $K_{i_n}^{(n)}$ along $(\overline{K}_{i_n}^{(n)}-K_{i_n}^{(n)})\frown R$. Restrict g on $K_{i_n}^{(n)}$ and next extend it to $\widehat{K}_{i_n}^{(n)}$ in the symmetric manner. We denote by $\hat{g}$ the above function thus obtained from g. It is clear that $\hat{g}$ belongs to $M_\Delta(\widehat{K}_{i_n}^{(n)})$ and $\hat{g}\geq c>0$ on $\widehat{K}_{i_n}^{(n)}$. Since $M_\Delta(\widehat{K}_{i_n}^{(n)})$ is an ideal of $M(\widehat{K}_{i_n}^{(n)})$ and g is inversible in $M(\widehat{K}_{i_n}^{(n)})$, $1=\hat{g}/\hat{g}$ belongs to $M_\Delta(\widehat{K}_{i_n}^{(n)})$. This shows that $\widehat{K}_{i_n}^{(n)}$ is a parabolic Riemann surface (cf. [3] and [1]). On the other hand, (a) implies that $\widehat{K}_{i_n}^{(n)}$ is a hyperbolic Riemann surface. This is absurd. Thus (a) implies (b).

(b) implies (c). In fact, $\tilde{g}(p, p_0)=\min(g(p, p_0), 1)$ belongs to $M_\Delta(R)$ and so $\tilde{g}(p, p_0)$ vanishes on Δ. Hence the same is true for $g(p, p_0)$. Since $E_k\frown\Delta$ is not empty, we get (c).

(c) implies (a). To show this we may clearly assume that p_0 is in R_1. We put $m=\inf (g(p, p_0); p\in(\overline{K}_{i_n}^{(n)}-K_{i_n}^{(n)})\frown R)>0$ and $u_0(p)=m^{-1}g(p, p_0)$. Then u_0 belongs to $\mathfrak{U}^{K_{i_n}^{(n)}}$ and so

$$w(p; K_{i_n}^{(n)})\leq u_0(p).$$

As (c) holds, so $\inf (w(p; K_{i_n}^{(n)}); p\in K_{i_n}^{(n)})=0$. Thus $w(p; K_{i_n}^{(n)})$ is not constant for all n. Hence we get the validity of (a). Q.E.D.

4. Next we consider the distribution of non-harmonic boundary points in ∂R. The following shows that the situation is very complicated and somewhat pathological in the viewpoint of our intuition.

Theorem 3. *Let F be a compact set in ∂R such that there exists a sequence $\{D_n\}_1^\infty$ of open sets in R satisfying $D_n\supset\overline{D}_{n+1}\frown R$, $n=1, 2,$*

$\cdots$, *and* $F=\bigcap_1^\infty \overline{D}_n$. *Then* $F\frown(\partial R-\Delta)$ *is non-empty.*

Proof. Let $U_n=D_n-\overline{D}_{n+1}\frown R$, $n=1,2,\cdots$. In U_n we take two simply connected Jordan domains $V_{n,0}$ and $V_{n,1}$ and a point p_n such that $p_n\in V_{n,1}\subset\overline{V}_{n,1}\subset V_{n,0}\subset\overline{V}_{n,0}\subset U_n$ and the annulus $A_n=V_{n,0}-\overline{V}_{n,1}$ is conformally equivalent to the annulus $(1<|z|<\exp(2^n\pi))$. Let $w_n(p)$ be a continuous function defined on R as follows:

$$w_n(p)=\begin{cases}0 & \text{on } R-V_{n,0};\\ \text{harmonic} & \text{on } A_n;\\ 1 & \text{on } \overline{V}_{n,1}.\end{cases}$$

Then from $\iint_R dw_n\wedge{}^*dw_n=2\pi/\text{mod } A_n$ and $\text{mod } A_n=\text{mod}(1<|z|<\exp(2^n\pi))=2^n\pi$, we get

$$\iint_R dw_n\wedge{}^*dw_n=2^{-(n-1)}.$$

Now we put

$$\varphi_n(p)=\sum_{i=1}^{n} w_i(p)$$

and

$$\varphi(p)=\sum_{i=1}^{\infty} w_i(p)$$

respectively. Clearly $\varphi_n\in M_0(R)$ and $\{\varphi_n\}_1^\infty$ is bounded and converges to φ uniformly on each compact subset of R and

$$\iint_R d(\varphi-\varphi_n)\wedge{}^*d(\varphi-\varphi_n)=\sum_{i=n+1}^{\infty}\iint_R dw_i\wedge{}^*dw_i=2^{-n}.$$

Hence $\{\varphi_n\}$ converges to φ in BD-convergence topology and so $\varphi\in M_\Delta(R)$. Let p_0 be an accumulation point of $\{p_n\}_1^\infty$. As p_0 lies in each of $\overline{D}_n$, so p_0 belongs to F. Since $\varphi(p_n)=w_n(p_n)=1$, we conclude that $\varphi(p_0)=1$. This shows that $p_0\notin\Delta$ or $p_0\in F\frown(\partial R-\Delta)$. Thus $F\frown(\partial R-\Delta)$ is non-empty. Q.E.D.

Corollary 3.1. *The harmonic boundary* Δ *is nowhere dense in* ∂R.

Proof. We have to show that for any $p_0\in\Delta$ and for any open neighborhood U of p_0, $U\frown(\partial R-\Delta)$ is non-empty. For this aim, we take an open neighborhood V of p_0 such that $\overline{V}\subset U$. Let $\{R_n\}$ be an exhaustion of R. Choosing a suitable subsequence of $\{R_n\}$, we may assume that $(R_{n+1}-\overline{R}_n)\frown V$, $n=1,2,\cdots$, are not empty. Then, by taking $F=V\frown\partial R$ and $D_n=V\frown(R-\overline{R}_n)$ in Theorem 3, $F\frown(\partial R-\Delta)$ is non-empty and so $U\frown(\partial R-\Delta)$ is non-empty. Q.E.D.

Corollary 3.2. *For each end* E_k, $E_k\frown(\partial R-\Delta)$ *is non-empty.*

Proof. We have proved that $E_k=\bigcap_1^\infty \overline{K}_{i_n}^{(n)}$ (cf. (1)). By putting $F=E_k$ and $D_n=K_{i_n}^{(n)}$, the assumption in Theorem 3 is satisfied. Hence $E_k\frown(\partial R-\Delta)$ is non-empty. Q.E.D.

5. In his paper [3], Royden asked whether or not a hyperbolic end can contain both points of Δ and $\partial R-\Delta$. By Theorem 2 and Corollary 3.2, this can be positively answered, that is, *any hyperbolic end contains both points of Δ and $\partial R-\Delta$.*

6. Although R^* is separable, it does not satisfy 2nd countability axiom. Namely,

Theorem 4. *No point of ∂R has a countable base of neighborhood system in R^*.*

Proof. Contrary to the assertion, suppose that a point p_0 in ∂R has a countable base of neighborhood system $\{U_n\}_1^\infty$. We may assume that U_n, $n=1,2,\cdots$, are open and $\overline{U}_{n+1}\subset U_n$, $n=1,2,\cdots$. We take an annulus A_n in $U_n-\overline{U}_{n+1}$ and construct the function $\varphi(p)$ on R as in the proof of Theorem 3. As φ is in $M(R)$, so it is continuous on R^* and a fortiori at p_0. Let p_n (resp. q_n) be a point in the interior (resp. exterior) boundary of A_n. Clearly $\{p_n\}_1^\infty$ and $\{q_n\}_1^\infty$ converge to p_0 respectively. By the continuity of φ at p_0,

$$\varphi(p_0)=\lim_n\varphi(p_n)=\lim_n w_n(p_n)=1$$

and at the same time

$$\varphi(p_0)=\lim_n\varphi(q_n)=\lim_n w_n(q_n)=0.$$

This is absurd. Thus no point in ∂R has a countable base of neighborhood system in R^*. Q.E.D.

Corollary 4.1. *Royden's compactification R^* is not metrizable.*

Corollary 4.2. *No point of ∂R is isolated in ∂R.*

Proof. Assume that p_0 is an isolated point in ∂R. Then $\{p_0\}$ is a component E_k of ∂R. Then, by (2), there exists a determining sequence $\{K_{i_n}^{(n)}\}$ of E_k such that $\bigcap_1^\infty\overline{K}_{i_n}^{(n)}=E_k$. Clearly $V_n=\{p_0\}\cup K_{i_n}^{(n)}$ is a neighborhood of p_0 in R^* and $\{V_n\}_1^\infty$ forms a base of neighborhood system of p_0. This is impossible in view of Theorem 4. Q.E.D.

Corollary 4.3. *Any end is non-degenerate in R^*.*

The last fact shows that there can exist a non-degenerate continuum in ∂R whose points are all irregular points for Dirichlet problem considered in the class of Dirichlet-finite harmonic functions.

References

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