131. On Quasi-normed Spaces over Fields with Non-archimedean Valuation

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The normed spaces over the fields with non-archimedean valuation were established by A. F. Monna [1]. In this paper, we shall consider the quasi-normed spaces over the fields with non-archimedean valuation.

Let K be a complete field with a non-archimedean valuation $|\lambda|$. We shall fix this field K throughout this paper.

1. General properties. **Definition 1.1.** *Let E be a linear space over a field K. An application $\|x\|$ of x is called a non-archimedean (n.a.) quasi-norm with the power r if it satisfies the following axioms:*

1. $\|x\|=0$ *if and only if* $x=\theta$.
2. $\|x+y\|\leq\max(\|x\|, \|y\|)$ *for all* $x, y\in E$.
3. $\|\lambda x\|=|\lambda|^r\|x\|$ *for* $\lambda\in K$ *and* $x\in E$, (r *real* $0<r<\infty$).

Let $\|x\|$ be a n.a. quasi-norm with the power r and let $d(x, y)=\|x-y\|$, $x\in E$, $y\in E$ then d is the distance on E. A linear topological space which is defined by the distance d is called a n.a. quasi-normed space with the power r.

Definition 1.2. *Let E be a n.a. quasi-normed space with the power r and if E is complete with the distance d, E will be called a n.a. (QN) space with the power r.*

We can prove the usual properties of quasi-normed spaces in n.a. quasi-normed spaces by the same ways [2–4] and [5]. Therefore we have the following theorems.

Theorem 1.1. *Let E be a n.a. (QN) space with the power r and N a closed subspace, then the quotient space E/N is a n.a. (QN) space with the power r.*

Theorem 1.2. *If E is a n.a. quasi-normed space with the power r then the space may be regarded as a dense subspace of a n.a. (QN) space $\hat{E}$ with the power r.*

We omit the proofs of the general theorems since they are proved by the same way as the archimedean case.

2. Linear transformations. Let E, F be two n.a. quasi-normed spaces with powers r, s and T a linear transformation which maps E into F.

Theorem 2.1. *A linear transformation T is continuous if and only if there exists a positive number M for which the following inequality holds:*

$$\|T(x)\|_s \leq M \|x\|_r^{s/r}.$$

Let $\mathcal{L}(E, F)$ be the set of continuous linear transformations which map E into F, these being two n.a. quasi-normed spaces with powers r, s then $\mathcal{L}(E, F)$ is a linear space. The application $T \to \|T\|$ is a n.a. quasi-norm in $\mathcal{L}(E, F)$ and $\mathcal{L}(E, F)$ is a n.a. (QN) space.

3. Quasi convex sets. **Definition 3.1.** *Let E be a linear space over a field K. An application $p(x)$ is called a n.a. quasi-semi-norm with the power r if it satisfies the following axioms*:

1. $p(x+y) \leq \max(P(x), p(y))$ *for all* $x, y \in E$.
2. $p(\lambda x) = |\lambda|^r p(x)$ *for* $\lambda \in K$ *and* $x \in E$, *(r real and* $0 < r < \infty$).

Suppose that there exists a n.a. quasi-semi-norm $p(x)$ with the power r in E. The set $A = \{x \in E; p(x) < 1\}$ has the following properties:

1. A is symmetric.
2. Let $x, y \in A$, we have $\lambda^s x + \mu^s y \in A$ for all $\lambda, \mu \in K$ such that $|\lambda| \leq 1$, $|\mu| \leq 1$, where s is $1/r$.
3. For any α, β such that $|\alpha| < |\beta|$, we have $\alpha^s A + \beta^s A \subset \beta^s A$.
4. For any α, β such that $|\alpha| < |\beta|$, we have $\alpha A \subseteq \beta A$.
5. For any α, $\alpha \neq 0$, the set αA is equivalent to the set of element $x \in E$ such that $p(x) < |\alpha|^r$.
6. A is absorbant: For any $y \in E$ there exists $\alpha > 0$ such that $y \in \lambda A$ for all $\lambda \in K$ satisfying $|\lambda| \geq \alpha$.
7. We shall consider the norm determined by A. Let $y \in E$ be given and consider a set of $\lambda \in K$ such that $y \in \lambda^s A$. The set is not empty. Let $\inf_{y \in \lambda^s A} |\lambda| = C$. We have the following relation on C and $p(y)$.

I. The valuation of K is dense.

For any λ such that $|\lambda| > C$, we have $y \in \lambda^s A$ and $p(y) < |\lambda|$. Since the valuation is dense we have $p(y) \leq C$. $C = 0$ implies $p(y) = 0$. Suppose $C \neq 0$, for any μ such that $|\mu| < C$, we have $y \notin \mu^s A$. Therefore $p(y) \geq |\mu|$ and $p(y) \geq C$. We have $p(y) = C$.

II. The valuation of K is discrete.

Suppose that the valuation of elements of K is defined by the power of number $\rho > 1$. Since the valuation of K is discrete, there exists $\lambda_0 \in K$ such that $C = |\lambda_0|$. Let $|\lambda_0| = \rho^k$. For any λ such that $|\lambda| = |\lambda_0|$, we have $y \in \lambda^s A$. Therefore $p(y) \leq |\lambda_0|$. $C = 0$ implies $p(y) = 0$. Suppose $C \neq 0$, for any μ such that $|\mu| < |\lambda_0|$ we have $y \notin \mu^s A$. Therefore

$$p(y) \geq |\mu|, \quad \text{and} \quad p(y) \geq \rho^{k-1},$$

we have

$$\rho^k > p(y) \geq \rho^{k-1}.$$

Definition 3.2. (1) *A subset A of a linear space E over the field K has the property (K^s) if $x \in A$, $y \in A$ implies $\lambda^s x + \mu^s y \in A$ for* *all* $|\lambda|\leq 1$, $|\mu|\leq 1$.

(2) *A subset A in E is called a quasi-K^s-convex, either A has the property (K^s) or A is denoted by the form x_0+A^* whenever A^* has the property (K^s) and x_0 is a fixed vector.*

(3) *A subset A in E is called an absorbant quasi-k^s-convex, either A has the property (K^s) and is absorbant or A is denoted by the form x_0+A^* whenever A^* has the property (K^s) and is absorbant and x_0 is a fixed vector.*

(4) *Let E be a linear space over the field K. The linear space is called a locally quasi-K^s-convex (topological) space if there exists a base of symmetric and absorbant quasi-K^s-convex sets.*

Theorem 3.1. *Suppose that there exists a n.a. quasi-semi-norm $p(x)$ with the power r in E. A topology defined by $p(x)$ is a locally quasi-K^s-convex topology, where s is $1/r$.*

Proof. Let $\mathscr{B}$ be a collection of $V_{|\lambda|}=p^{-1}([0,|\lambda|])=\{x\in E;\ p(x)\leq|\lambda|,\ \lambda\in K\}$. We shall show that $\mathscr{B}$ is a base for E. To prove it, for $\mu\neq 0$, we have

$$\begin{aligned}\mu V_{|\lambda|}&=\{\mu x;\ p(x)\leq|\lambda|\}\\&=\left\{y;\ p\left(\frac{1}{\mu}y\right)\leq|\lambda|\right\}\\&=\{y;\ p(y)\leq|\mu|^r|\lambda|\}\\&=V_{|\mu|^r|\lambda|}.\end{aligned}$$

Let $x\in\mu V_{|\lambda|}$ and $|\mu|\leq 1$, then

$$p(x)\leq|\mu|^r|\lambda|\leq|\lambda|$$

and

$$x\in V_{|\lambda|}.$$

Therefore $\mu V_{|\lambda|}\subset V_{|\lambda|}$ for all μ such that $|\mu|\leq 1$. On the other hand, let $\alpha=\left(\frac{1}{|\lambda|}p(x)\right)^s$, for μ such that $|\mu|\geq\alpha$

$$\begin{aligned}&p(x)^s=\alpha|\lambda|^s\leq|\mu|\,|\lambda|^s\\&p(x)\leq|\mu|^r|\lambda|\end{aligned}$$

and

$$x\in\mu V_{|\lambda|}.$$

Therefore, for any $x\in E$, there exists a positive number α and for all μ such that $|\mu|\geq\alpha$, $x\in\mu V_{|\lambda|}$.

From the following relation,

$$\begin{aligned}\alpha V_{\frac{|\lambda|}{|\alpha|^r}}&=\left\{\alpha x;\ p(x)\leq\frac{|\lambda|}{|\alpha|^r}\right\}=\left\{y;\ p\left(\frac{1}{\alpha}y\right)\leq\frac{|\lambda|}{|\alpha|^r}\right\}\\&=\{y;\ p(y)\leq|\lambda|\}=V_{|\lambda|},\end{aligned}$$

we have, for any $V\in\mathscr{B}$ and $\alpha\in(K-0)$, $\alpha V\in\mathscr{B}$.

Moreover, $V_{|\lambda|}+V_{|\lambda|}\subset V_{|\lambda|}$.

Next, we shall show that $V_{|\lambda|}$ is quasi-K^s-convex set. Let $x, y\in V_{|\lambda|}$ and λ,μ be $|\lambda|\leq 1$, $|\mu|\leq 1$

$$p(\lambda^s x+\mu^s y)\leq \max(p(\lambda^s x),\ p(\mu^s y))\leq|\lambda|.$$

Therefore $\mathscr{B}$ is a collection of symmetric and absorbant quasi-K^s-convex sets. The topology by $p(x)$ is a locally quasi-K^s-convex topology.

References

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