8. A Remark on Regular Semigroups

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A *semigroup* is a non-empty set which is closed with respect to an associative binary multiplication. A *left ideal* L of S is a non-empty subset of S such that $SL\subset L$. A *right ideal* R of S is a non-empty subset of S such that $RS\subset R$. A *two-sided ideal* or *ideal* of S is a subset which is both a left and a right ideal. If a is an element of the semigroup S, (a) denotes the smallest left ideal of S containing a. A left ideal L of S is called *principal* if and only if $L=(a)_L$ for some a in S. Similarly we can define principal right ideal $(a)_R$ and principal two-sided ideal (a).

The concept of regular ring was introduced by J. von Neumann [5] as follows: an arbitrary (associative) ring A is called *regular* if to any element a of A there exists an x in A such that $axa=a$. The concept of regular semigroup is defined analogously (see e.g. [1]). L. Kovács [3] characterized the regular rings as rings satisfying the property:

$$R\cap L=RL,$$

for every right ideal R and every left ideal L of A. K. Iséki [2] extended this characterization to semigroups. In this connection we prove the following

Theorem 1. *To any semigroup S the following conditions are equivalent*:

1) *S is regular,*

2) *$R\cap L=RL$, for every right ideal R and every left ideal L of S,*

3) *$(a)_R\cap(b)_L=(a)_R(b)_L$, for every pair of elements a,b in S,*

4) *$(a)_R\cap(a)_L=(a)_R(a)_L$, for every element a of S.*

Following J. A. Green [1] we shall say that an element a of a semigroup S is *regular* if and only if there exists $x\in S$ so that $axa=a$. First we prove that *an element a of a semigroup S is regular if and only if the condition* 4) *of Theorem* 1 *holds.* Let a be regular. Then by Lemma 3 of [4], $(a)_L=(e)_L$ and $(a)_R=(f)_R$, where e,f are idempotent elements. Let u be an element of $(a)_R\cap(a)_L=fS\cap Se$. Then $u=fs=s'e$. This implies that $u=fu=fs'e=ffs'e\in(fS)(Se)=(a)_R\cdot(a)_L$, therefore

$$(a)_R\cap(a)_L\subseteq(a)_R(a)_L.$$

The converse is trivial, that is the condition 4) holds.

Conversely, let us suppose that condition 4) holds. Clearly

$a \in (a)_R \cap (a)_L = (a)_R(a)_L$. Since $(a)_R = a \cup aS$ and $(a)_L = a \cup Sa$, it follows that

$$a \in (a)_R(a)_L = a^2 \cup aSa \cup aS^2a \subset a^2 \cup aSa.$$

Thus either $a = a^2$ or $a \in aSa$, i.e. a is regular in both cases.

Combining this result with Lemma 3 of [4] we obtain

Theorem 2. *The following four propositions concerning an element a of a semigroup S are equivalent:*

1) *a is regular,*
2) *$(a)_L = (e)_L$ for an idempotent element e of S,*
3) *$(a)_R = (f)_R$ for an idempotent element f of S,*
4) *$(a)_R \cap (a)_L = (a)_R \cdot (a)_L$.*

Proof of Theorem 1. 1) implies 2): This is known (see: Iséki [2]). That 2) implies 3) and 3) implies 4) it is evident. 4) implies 1): This follows from Theorem 2, since the regular semigroup can be defined as semigroup every element of which is regular. From Theorem 1 we conclude the following

Corollary. *A commutative semigroup is regular if and only if every principal ideal of it is idempotent.*

References

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