6. *A Certain Type of Vector Field. I*

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I. Let M be a differentiable manifold of class C^3 and of dimension n and assume that M has an affine connection. We denote covariant differentiation by $X \in T_x$ with respect to this affine connection by ∇_X provided that T_x is the tangent vector space at $x \in M$. Then to each vector field V defined in a neighborhood of x there is attached a homomorphism A_V of T_x into itself, provided that for $X \in T_x$, $A_V(X)$ is defined as usual to be $-\nabla_X V$.

It is expected that there will exist certain correspondences between the geometric natures of vector field V and the algebraic natures of A_V. The objective of the present paper is to study these correspondences in a certain case which will be stated below and to arrange the preliminaries of the author's previous paper [1][1] which has determined both the metrics and the topological types of the complete manifolds admitting a torse-forming vector field with some singularities. It is interesting to note that these manifolds show remarkable similarities, both in their metric aspects and in their topological aspects, to the hypersurfaces of rotation which admit at least one torse-forming vector field as is shown in the sequel.

Let f be a differentiable function defined in a neighborhood of x and consider the germ defined by f which will be denoted by $[f]$. The total of $[f]$ forms a ring, denoted by $\mathfrak{S}$. Let $\mathfrak{S}[\eta]$ denote a polynomial ring over coefficient ring $\mathfrak{S}$. Further let Grad be a linear map of module $\mathfrak{S}$ into T_x^* which to each $[f]$ assigns the value at x of the gradient covector of f.

If M has a Riemann metric $g(X, Y)$ $(X, Y \in T_x)$, we can define a linear map $P_Z (Z \in T_x)$ by $P_Z(X) = g(X, Z)/g(Z, Z)Z$. Then one of the simplest types of A_V is the one with $A_V \in \mathfrak{S}[P_V]$.[2] If we write

$$\mathfrak{S}^* = \{[f] \mid P_V \circ \mathrm{Grad}\,[f] = \mathrm{Grad}\,[f]\},$$

then $\mathfrak{S}^*$ is a subring of $\mathfrak{S}$ and if R denotes the real number field, then R also is a subring of $\mathfrak{S}$ in the natural sense. Let $\mathfrak{P}$ denote a specialization: $\mathfrak{S}[\eta] \ni s(\eta) \to s(0) \in \mathfrak{S}$. Then $\mathfrak{P}^{-1}(\mathfrak{S}^*)$ is a subring of $\mathfrak{S}[\eta]$ or of $\mathfrak{S}[P_V]$ when η is regarded as P_V and, similarly, $\mathfrak{P}^{-1}(R)$ is a subring of them.

The vector fields discussed in the present paper are the gradient

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1) Numbers in brackets refer to the reference at the end of the paper.

2) More precisely, $A_v = i_x(f(P_v))$ for some $f \in \mathfrak{S}[\eta]$, where i_x means a map assigning the value at x of s to each $[s]$.

ones of the above-mentioned kind, especially the ones with $A_V \in \mathfrak{P}^{-1}(\mathfrak{S}^*)$. This kind of vector field, what is called a torse-forming vector field, has been investigated by S. Sasaki, K. Yano and other Japanese mathematicians from another angle of the theory. It is worth while to mention that this kind of vector field is very closely related to the theory of relativity.

The main theorems in the present paper are as follows.

Theorem A. For a vector field V, A_V is a symmetric operator and belongs to $\mathfrak{P}^{-1}(\mathfrak{S}^*)$ in a neighborhood of a point of M if and only if the following three conditions are satisfied: 1) For movement in the transversal directions to V, there exists a fixed point in the direction indicated by V. 2) V has a potential. 3) The trajectories of V are geodesics, provided that V is assumed not to be a parallel vector field.

Theorem B. Let $D'=1/g(V, V)\, D$, where D is the usual differential operator acting on $\mathfrak{S}[\eta]$. Then there exists a vector field V for which A_V is symmetric and belongs to $\mathfrak{P}^{-1}(R)\frown D'^{-1}(R)$ in a neighborhood of a point of M if and only if the metric form of M, by taking a suitable coordinate system, can be expressed in one of the following forms: 1) $ds^2=\sinh^2(cx^n+d)ds_0^2+(dx^n)^2$ 2) $ds^2=\sin^2(cx^n+d)ds_0^2+(dx^n)^2$ 3) $ds^2=(x^n)^2ds_0^2+(dx^n)^2$,
where c and d in 1) and 2) are constants and x^n is the n-th coordinate of x and ds_0^2 in each expression is a suitable metric form of dimension $n-1$ that is independent of x^n.

Corollary. If M is an Einstein space, $A_V \in \mathfrak{S}[P_V]$ implies $A_{\sigma V} \in \mathfrak{P}^{-1}(R)\frown D'^{-1}(R)$, where σ is a suitable function.

Although the above theorems still hold in a Finsler manifold, some convections on terminology become necessary.

II. In this section we shall sketch the outline of a new global theory of Finsler manifolds and fix the ways of using certain terminology.

Let M be a differentiable manifold satisfying the second axiom of separability and let $\mathfrak{B}$ be a principal bundle with base space M and structural group $O(n)$, where $n=\dim M$. Then $O(n)$ acts on the total space B of $\mathfrak{B}$ from the right and consequently $O(n-1)$, considered to be a subgroup of $O(n)$ in a natural way, does so likewise. Then $\mathfrak{B}$ is called a Finsler bundle, if it satisfies these conditions:

1) An $O(n-1)$-invariant Riemann metric $g(X, Y)$ ($X, Y \in T_u$: The tangent space at $u \in B$) is given on B.

2) A bundle connection ω is given on $\mathfrak{B}$ in such a way as $\omega_u^{-1}(0)$ $(=\omega^{-1}(0)\frown T_u)$ is orthogonal to the fibre through $u \in B$.

3) There exist a principal bundle $\mathfrak{B}^*$ over B with structural group $O(n)$ and a bundle map $\bar{p}$ of $\mathfrak{B}^*$ onto $\mathfrak{B}$ with $p\circ\bar{p}=p\circ p^*$, where

p is the projection of $\mathfrak{B}$ and p^* is that of $\mathfrak{B}^*$. By the use of $\bar{p}$, each element of the group $O(n-1)$ acting on B can be extended to a bundle map of $\mathfrak{B}^*$ onto itself. Then any $O(n-1)$-invariant bundle connection on $\mathfrak{B}^*$ is termed a Finsler connection on $\mathfrak{B}$.

Now consider a more restricted type of Finsler bundle and connection. For that let $\mathfrak{B}$ be refined to the principal bundle associated to the tangent bundle $T(M)$ the structural group of which has been reduced to $O(n)$. Horizontal spaces $\omega^{-1}(0)$ determine a bundle $\mathfrak{B}'$ with base space B and group $O(n)$ which is a subbundle of an $O\left(n-\frac{n(n-1)}{2}\right)$-bundle $GL(n)$-equivalent to $T(B)$. Let $\mathfrak{B}''$ be the associated principal bundle of $\mathfrak{B}'$. It is almost clear that there exists the above-mentioned type of map p of $\mathfrak{B}''$ onto $\mathfrak{B}$. Moreover we easily see that there exists a map φ' with commutative

$$\begin{array}{ccccc} & \overset{p}{\nearrow} & M & \xrightarrow{\text{an identity}} & M \\ B & & \uparrow p' & & \uparrow p'' \\ & \underset{\varphi''}{\searrow} & B/O(n-1) & \xrightarrow{\varphi'} & T(M) \end{array}$$

where $B/O(n-1)\xrightarrow{p'}M$ is a bundle obtained from $\mathfrak{B}$ by dividing B by $O(n-1)$ and p'' is the projection of $T(M)$ and φ'' is the natural map of B onto $B/O(n-1)$. If we write $\varphi=\varphi'\circ\varphi''$, then φ is a fibre-preserving map of $\mathfrak{B}$ into $T(M)$. With this φ we can define a parallel displacement of line elements in this way: Let γ be a curve on B with end-point u. Then the parallel displacement along γ of line-element $\varphi(u)$, by definition, is $dp\circ\gamma(p_u^{-1}\varphi(u))$, provided that p_u is an isomorphism of $\omega_u^{-1}(0)$ onto $T_{p(u)}$ given by p.

As well known there exists a symmetric and metric connection induced by the Riemann metric $g(X, Y)$. It is hoped that the Finsler connection determined by E. Cartan would be obtained from this kind of connection by the orthogonal projection onto the horizontal spaces. Assume that a linear Finsler connection is given. Taking horizontal frames $(e_i')_{i=1,2,\cdots,n}$ at each point u of B, we have $de_i'=\omega_i^j e_j'$. We write $\omega_i^j=\omega_i^{*j}+\omega_i'^j$, where ω_i^{*j} is a differential form on the horizontal spaces and $\omega_i'^j$ one on the vertical spaces. Set $e_i=p_u(e_i')$. Then $de_i=(\omega_i^{*j}+\omega_i'^j)e_j$. In the case of Cartan's connection $\omega_i^{*j}=\Gamma_{ik}^{*j}dx^k$. In what follows we write $\nabla_Y^* X=(dX^i+X^j\omega_j^{*i})Ye_i$. Moreover if we denote by ϖ an absolute differential of a line element with respect to ω_i^{*j}, then $\omega_i'^j=A_{ik}^j\varpi^k$.

III. In this section a correspondence between the vectors and the covectors in Finsler manifolds will be treated of. Let T_x^* denote the dual space of tangent space T_x. In a Riemannian manifold there exists a canonical correspondence between T_x and T_x^*, but in a Finsler manifold it fails to appear without any restriction on vectors or covectors. Actually this correspondence given by assigning to each

$X \in T_x$ a linear map of T_x into R: $T_x \ni Y \to g(p_u^{-1}(X), p_u^{-1}(Y)) \in R$ can be defined only for some part of T_x which will be denoted by S_x in what follows. It is, nevertheless, this part of T_x that is of the most importance, particularly from the topological point of view. As is easily seen, $X \in T_x$ belongs to S_x if and only if

$$\partial_u g(p_u(X), p_u(Y)) = 0 \quad \text{for } u \text{ with } p(u) = x \text{ and } Y \in T_x.$$

IV. A generalization of Theorem A and Theorem B is as follows.

Theorem A and Theorem B hold if V is restricted to a vector field belonging to S_x at all the points in the neighborhood and ∇ is assumed as ∇^* in II.

V. It is well known that a torse-forming vector field is closely connected with the conformal nature of the manifold. Then it will be convenient that a general picture of conformal maps in Finsler geometry is given here for later use.

The present author thinks that there exist two kinds of conformal map for Finsler bundles geometrically, as is defined below.

1) A conformal map of the general kind is the usual one, that is a bundle map h of a Finsler bundle $\mathfrak{B}_1$ into another Finsler bundle $\mathfrak{B}_2$ satisfying this:

$$g_2(p_{h(u)}(dh(X)), p_{h(u)}(dh(Y))) = \exp(2\sigma) g_1(p_u(X), p_u(Y)),$$

where σ is an $O(n-1)$-invariant function defined on B_1, the total space of $\mathfrak{B}_1$, in general. In what follows, however, it is assumed $O(n)$-invariant and identified with a function on M_1, the base space of $\mathfrak{B}_1$.

2) A conformal map of the restricted kind, by definition, is a conformal map of the general kind with (the gradient covector of σ) $\in S_x$.

We shall show how the latter ones arise from geometric problems in the sequel.

VI. Let $A(x) = i_x \circ \mathfrak{P}(A_V)$. Then we have

Proposition. $A(x)$ is a differentiable function and $[A(x)] = \mathfrak{P}(A_V)$.

Proof. Let $A_V = f(P_V)$, where $f \in \mathfrak{S}[\eta]$. Let $P^\perp$ be the orthogonal projection into the transversal directions to V. Then

$$P^\perp \circ A_V = P^\perp \circ f(P_V) = A(x) P^\perp.$$

The differentiabilities of the left hand side and $P^\perp$ of the right hand side imply that of $A(x)$.

VII. A note on [1]: Condition (iv) in the introduction of [1] can be replaced by this one.

(iv′) A_V is never reduced to the null-operator.

This condition not only induces the mutual isolation of points with $V=0$, but adds strong restriction to the indices of V at such points, as has been shown in [1]. Therefore the author's previous paper [1] leaves something to be desired.

Reference

[1] T. Maebashi: Vector fields and space forms, J. Fac. Sci. Hokkaido Univ., **15**, 62–92 (1960).