53. On a Problem of C. D. Papakyriakopoulos

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1. A problem of C. D. Papakyriakopoulos. Let M be an orientable closed 3-manifold and let F_1 and F_2 be two orientable closed surfaces of the same genus h in M such that $M-F_1$ and $M-F_2$ consist of two components, the closure of each one of which being a solid torus of genus h. Then the *uniqueness problem* proposed by C. D. Papakyriakopoulos [1] is the following:

Does there exist a homeomorphism of M onto itself carrying F_1 onto F_2?

In this note we shall show that the problem is affirmative for $h=1$.

Theorem. *Let M be an orientable closed 3-manifold and let F_1 and F_2 be two orientable closed surfaces of genus one in M, such that $M-F_1$ and $M-F_2$ consist of two components, the closure of each one of which being a solid torus of genus one. Then there exists a homeomorphism of M onto itself carrying F_1 onto F_2.*

Before we proceed to the proof of the theorem, we shall prove the following lemma on a lens space.

Lemma. *Let M be a lens space of the type $L(p,q)$, where $0\leqq q\leqq \frac{p}{2}$ and $(p,q)=1$[1] [2]. Let T be a closed orientable surface of genus one in M, such that $M-T$ consists of two components, whose closures V and V' are solid tori of genus one. Then there exist meridians[2] m, m' and longitudes[2] l, l' of V and V' respectively, which satisfy the following condition:*

m is homologous to $qm'+pl'$ on $T=\partial V=\partial V'$ or

m' is homologous to $qm+pl$ on T.

Proof of Lemma. There exist meridians m, m' and longitudes l, l' of V and V' respectively, such that

$$\begin{pmatrix} m \\ l \end{pmatrix} \text{ is homologous to } \begin{pmatrix} q' & p \\ x & y \end{pmatrix}\begin{pmatrix} m' \\ l' \end{pmatrix} \text{ on } T,$$

where $0\leqq q'\leqq \frac{p}{2}$, $(p, q')=1$ and $\begin{vmatrix} q' & p \\ x & y \end{vmatrix}=1$.

Then we shall obtain $q=q'$ or $qq'\equiv\pm 1 \pmod p$.

1) Throughout this paper, if $p=1$, we consider $q=0$ and if $p=0$, we consider $q=1$.

2) A meridian m of a full torus V of genus one means an oriented simple closed curve on ∂V which is homotopic zero in V and a longitude of V means an oriented simple closed curve on ∂V which has the intersection number 1 with m.

If $q=q'$, Lemma is satisfied for (m, l) and (m', l').

If $qq'\equiv\pm 1 \pmod p$, then $\begin{pmatrix} m' \\ l' \end{pmatrix}$ is homologous to $\begin{pmatrix} y & -p \\ -x & q' \end{pmatrix}\begin{pmatrix} m \\ l \end{pmatrix}$ on T.

From the conditions $q'y\equiv 1 \pmod p$, $q'q\equiv\pm 1 \pmod p$ and $(q', p)=1$, we obtain $\pm q\equiv y \pmod p$, that is, $y=\pm q-tp$ for some integer t. Let us denote

$$\begin{pmatrix} \tilde{m} \\ \tilde{l} \end{pmatrix}=\begin{pmatrix} \pm 1 & 0 \\ -t & -1 \end{pmatrix}\begin{pmatrix} m \\ l \end{pmatrix},$$

then $\tilde{m}$ and $\tilde{l}$ are a meridian and a longitude of V respectively and m' is homologous to $q\tilde{m}+p\tilde{l}$ from the following calculation.

$$\begin{aligned}\begin{pmatrix} y & -p \\ -x & q' \end{pmatrix}\begin{pmatrix} m \\ l \end{pmatrix} &= \begin{pmatrix} \pm q-tp & -p \\ -x & q' \end{pmatrix}\begin{pmatrix} m \\ l \end{pmatrix} \\ &= \begin{pmatrix} \pm q-tp & -p \\ -x & q' \end{pmatrix}\begin{pmatrix} \pm 1 & 0 \\ -t & -1 \end{pmatrix}^{-1}\begin{pmatrix} \tilde{m} \\ \tilde{l} \end{pmatrix} \\ &= \begin{pmatrix} \pm q-tp & -p \\ -x & q' \end{pmatrix}\begin{pmatrix} \pm 1 & 0 \\ \mp t & -1 \end{pmatrix}\begin{pmatrix} \tilde{m} \\ \tilde{l} \end{pmatrix} \\ &= \begin{pmatrix} q & p \\ \mp x\mp q't & -q' \end{pmatrix}\begin{pmatrix} \tilde{m} \\ \tilde{l} \end{pmatrix}.\end{aligned}$$

Thus our Lemma is proved.

2. The proof of the theorem. We may suppose that M is a lens space of the type $L(p, q)$, where $0\leqq q\leqq \frac{p}{2}$ and $(p, q)=1$. Let V_i and V'_i be the closures of the two components of $M-F_i$ $(i=1, 2)$. From the above lemma there exist meridians m_i, m'_i $(i=1, 2)$ and longitudes l_i, l'_i $(i=1, 2)$ of V and V' respectively which satisfy the following conditions:

m_1 is homologous to $qm'_1+pl'_1$ on $\partial V'_1=F_1$ or m'_1 is homologous to qm_1+pl_1 on $\partial V_1=F_1$ and

m_2 is homologous to $qm'_2+pl'_2$ on $\partial V'_2=F_2$ or m'_2 is homologous to qm_2+pl_2 on $\partial V_2=F_2$.

Without loss of generality, we may suppose that $\begin{pmatrix} m_i \\ l_i \end{pmatrix}$ is homologous to $\begin{pmatrix} q & p \\ x_i & y_i \end{pmatrix}\begin{pmatrix} m'_i \\ l'_i \end{pmatrix}$ on F_i $(i=1, 2)$. From the fact $\begin{vmatrix} q & p \\ x_i & y_i \end{vmatrix}=\pm 1$ and $(p, q)=1$, there exists an integer t which satisfies $x_1=x_2+tq$ and $y_1=y_2+tp$.

Let k be a homeomorphism from V'_1 to V'_2 which carries m'_1 and l'_1 to m'_2 and l'_2 respectively.

Let us suppose that M is constructed by the identification f_i of the boundary ∂V_i of V_i with the boundary $\partial V'_i$ of V'_i $(i=1, 2)$.

Let h' be the homeomorphism $f_2^{-1}kf_1$ from ∂V_1 to ∂V_2.

$$\text{Then } \begin{pmatrix} h'(m_1) \\ h'(l_1) \end{pmatrix} \text{ is homologous to } \begin{pmatrix} q & p \\ x_1 & y_1 \end{pmatrix} \begin{pmatrix} q & p \\ x_2 & y_2 \end{pmatrix}^{-1} \begin{pmatrix} m_2 \\ l_2 \end{pmatrix}$$

$$= \begin{pmatrix} q & p \\ x_2+qt & y_2+pt \end{pmatrix} \begin{pmatrix} q & p \\ x_2 & y_2 \end{pmatrix}^{-1} \begin{pmatrix} m_2 \\ l_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} q & p \\ x_2 & y_2 \end{pmatrix} \begin{pmatrix} q & p \\ x_2 & y_2 \end{pmatrix}^{-1} \begin{pmatrix} m_2 \\ l_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} m_2 \\ l_2 \end{pmatrix} \text{ on } \partial V_1.$$

Therefore the homeomorphism h' from ∂V_1 to ∂V_2 may be extended to a homeomorphism h from V_1 to V_2.

Then it is clear that the following diagram is commutative on the boundary of V_i and V_i' $(i=1, 2)$.

$$\begin{array}{ccc} V_1 & \xrightarrow{f_1} & V_1' \\ h \downarrow & & \downarrow k \\ V_2 & \xrightarrow{f_2} & V_2'. \end{array}$$

Defining a homeomorphism f from M to M by

$$\begin{aligned} f(x) &= h(x) \quad \text{if } x \in V_1 \\ &= k(x) \quad \text{if } x \in V_1', \end{aligned}$$

we obtain a homeomorphism from M to M which carries F_1 to F_2. Thus our Theorem is proved.

References

[1] C. D. Papakyriakopoulos: Some problems on 3-dimensional manifolds, Bull. Amer. Math. Soc., **64**, 317–335 (1958).

[2] H. Seifert und W. Threlfall: Lehrbuch der Topologie, Leipzig (1934).