48. On the Definition of the Cross and Whitehead Products in the Axiomatic Homotopy Theory. II

By Ryōsuke NAKAGAWA
Department of Mathematics, Tokyo University of Education
(Comm. by K. KUNUGI, M.J.A., April 12, 1961)

1. Introduction. In the preceding paper I, we have described the definition of the cross and Whitehead products. In this paper we shall show a few properties of the cross and Whitehead products as consequences of their definition and prove the existence and uniqueness of these products.

2. Immediate consequences from the axiom. Consider two maps $f:(X,x_0)\to(X',x_0')$ and $g:(Y,y_0)\to(Y',y_0')$ and let $f\times g:(X\times Y, X\vee Y, (x_0,y_0))\to(X'\times Y', X'\vee Y', (x_0',y_0'))$ be a map defined by $(f\times g)(x,y)=(f(x), g(y))$.

Proposition 1. *For* $\alpha\in\pi_m(X,x_0)$, $\beta\in\pi_n(Y,y_0)$, *we have*

$$(f\times g)_\#(\alpha\times\beta)=f_\#\alpha\times g_\#\beta.$$

This is easily proved and the proof is omitted.

Now let $\tau: X\times Y\to Y\times X$ be a map such that $\tau(x,y)=(y,x)$.

Proposition 2. *For* $\alpha\in\pi_m(X,x_0)$, $\beta\in\pi_n(Y,y_0)$, *we have*

$$\tau_\#(\alpha\times\beta)=(-1)^{mn}(\beta\times\alpha). \qquad (1)$$

In order to prove this, we shall need the following lemma, whose proof will be omitted.

Lemma 3. *Let $f, g:(X,x_0)\to(Y,y_0)$ be H-homomorphisms between H-spaces X and Y with units x_0, y_0 respectively. An H-homomorphism $h=f\cdot g:(X,x_0)\to(Y,y_0)$ is defined by $h(x)=f(x)\cdot g(x)$, $x\in X$. Then we have $h_\#(\alpha)=f_\#(\alpha)+g_\#(\alpha)$, for $\alpha\in\pi_n(X,x_0)$, $n>0$. If X and Y are loop spaces, $\pi_0(X,x_0)$ and $\pi_0(Y,y_0)$ may be considered as groups. In this case the above relation holds also.*

Proof of Prop. 2. In cases $m=n=0$; $m=0$, $n>0$; $m>0$, $n=0$, we can show directly by definition that the formula (1) holds. Now we assume that the formula (1) holds for $k<m$, $l<n$. Let $\Omega\tau: \Omega X\times\Omega Y\to\Omega Y\times\Omega X$, $\tau': X\vee Y\to Y\vee X$ and $\Omega\tau':\Omega(X\vee Y)\to\Omega(Y\vee X)$ be maps induced by τ. Then $\Omega\partial\tau_\#(\alpha\times\beta)=\Omega\tau'_\#\partial(\alpha\times\beta)=(\Omega\tau')_\#\Omega\partial(\alpha\times\beta)$ $=(-1)^{n-1}(\Omega\tau')_\#\varphi_\natural(\Omega\alpha\times\Omega\beta)=(-1)^{n-1}((\Omega\tau')\circ\varphi)_\natural(\Omega\alpha\times\Omega\beta)$. A map $(\Omega\tau')\circ\varphi:\Omega X\times\Omega Y\to\Omega(Y\vee X)$ is defined by $((\Omega\tau')\circ\varphi)(x,y)=(y_0,x)(y,x_0)(y_0,x^{-1})(y^{-1},x_0)$. On the other hand, $(\varphi\circ(\Omega\tau))(x,y)=(y,x_0)(y_0,x)(y^{-1},x_0)(y_0,x^{-1})$. Therefore $(\Omega\tau')\circ\varphi=(\varphi\circ(\Omega\tau))^{-1}$. By Lemma 3, we have $((\Omega\tau')\circ\varphi)_\natural$ $=-(\varphi\circ(\Omega\tau))_\natural$. Hence $\Omega\partial\tau_\#(\alpha\times\beta)=(-1)^n(\varphi\circ(\Omega\tau))_\natural(\Omega\alpha\times\Omega\beta)=(-1)^n(\varphi_\natural\circ(\Omega\tau)_\#)(\Omega\alpha\times\Omega\beta)=(-1)^n\varphi_\natural((-1)^{(m-1)(n-1)}(\Omega\beta\times\Omega\alpha))=(-1)^{mn}\Omega\partial(\beta\times\alpha)$. Thus we have $\tau_\#(\alpha\times\beta)=(-1)^{mn}\beta\times\alpha$.

Corollary 4. *Let $f:(X,x_0)\to(X',x_0')$ and let $\alpha\in\pi_m(X,x_0)$, $\beta\in\pi_n(X,x_0)$. Then we have the following formulas*:

$$f_\#[\alpha,\beta]=[f_\#\alpha,f_\#\beta],$$
$$[\alpha,\beta]=(-1)^{mn}[\beta,\alpha].$$

3. Existence and uniqueness. In this section we shall construct a cross product operation in a given axiomatic homotopy theory $H=\{\pi,\#,\partial\}$ and show its uniqueness.

Let $\alpha\in\pi_m(X,x_0)$, $\beta\in\pi_n(Y,y_0)$. Then $\alpha\times\beta$ is well defined for m or $n=0$ by definition. Now let $m>0$, $n>0$ and assume that $\alpha\times\beta$ is already constructed for each $k<m$ and $l<n$. Then we have $\Omega^{-1}\varphi_\natural(\Omega\alpha\times\Omega\beta)\in\pi_{m+n-1}(X\vee Y,(x_0,y_0))$ and if we show that $\Omega^{-1}\varphi_\natural(\Omega\alpha\times\Omega\beta)\in$ Image $\partial:\pi_{m+n}(X\times Y, X\vee Y,(x_0,y_0))\to\pi_{m+n-1}(X\vee Y,(x_0,y_0))$, we can construct $\alpha\times\beta$ such that the axiom is satisfied. Let $p:X\times Y\to X$, $q:X\times Y\to Y$ be projections and let $\bar p:X\vee Y\to X$, $\bar q:X\vee Y\to Y$ be restricted maps of p and q respectively.

Lemma 5. *Let $n>1$. A boundary homomorphism $\partial:\pi_n(X\times Y, X\vee Y,(x_0,y_0))\to\pi_{n-1}(X\vee Y,(x_0,y_0))$ is a monomorphism. Moreover, we have Image ∂=Kernel $\bar\eta$, where $\bar\eta$ is a homomorphism of $\pi_{n-1}(X\vee Y,(x_0,y_0))$ to a direct product $\pi_{n-1}(X,x_0)\times\pi_{n-1}(Y,y_0)$ defined by $\bar\eta(\alpha)=(\bar p_\#(\alpha),\bar q_\#(\alpha))$.*

This lemma can be proved in the axiomatic homotopy theory and the proof is omitted.

Now let $\Omega\bar p:\Omega(X\vee Y)\to\Omega X$ be a map induced by $\bar p$. Then $\bar p_\#\Omega^{-1}\varphi_\natural(\Omega\alpha\times\Omega\beta)=\Omega^{-1}(\Omega\bar p)_\#\varphi_\natural(\Omega\alpha\times\Omega\beta)=\Omega^{-1}(\Omega\bar p\circ\varphi)_\natural(\Omega\alpha\times\Omega\beta)=0$, because $\Omega\bar p\circ\varphi\simeq 0$. Similarly, $\bar q_\#\Omega^{-1}\varphi_\natural(\Omega\alpha\times\Omega\beta)=0$. By Lemma 5, we have $\Omega^{-1}\varphi_\natural(\Omega\alpha\times\Omega\beta)\in$ Image ∂.

Since ∂ is a monomorphism, it is noted that the element $\alpha\times\beta$ satisfying the formula $\Omega\partial(\alpha\times\beta)=(-1)^{n-1}\varphi_\natural(\Omega\alpha\times\Omega\beta)$ is uniquely determined.

Two homotopy theories with cross products $H=\{\pi,\#,\partial,\times\}$, $\bar H=\{\bar\pi,\bar\#,\bar\partial,\bar\times\}$ are said to be equivalent if there exists an equivalence $h=\{h_n\}$ between $H=\{\pi,\#,\partial\}$ and $\bar H=\{\bar\pi,\bar\#,\bar\partial\}$ (cf. [2]) such that $h_{m+n}(\alpha\times\beta)=h_m\alpha\,\bar\times\, h_n\beta$ for $\alpha\in\pi_m(X,x_0)$, $\beta\in\pi_n(Y,y_0)$.

It can be easily seen from the above considerations that any two homotopy theories with cross products are equivalent.

The existence and uniqueness of the Whitehead product operation can be obtained directly by the above results.

4. Usual cross products. To conclude this paper, we shall show that the usual cross product operation satisfies the conditions described in the preceding paper I, § 2.

We recall the definition of the cross products in the usual homotopy theory.[1] Let $f:(I^m, \dot{I}^m)\to(X, x_0)$ and $g:(I^n, \dot{I}^n)\to(Y, y_0)$ be representatives of $\alpha\in\pi_m(X, x_0)$ and $\beta\in\pi_n(Y, y_0)$ $(m>0,\ n>0)$ respectively. Define a map $h:(I^m\times I^n, (I^m\times I^n)^{\cdot})\to(X\times Y, X\vee Y)$ by $h(x, y)=(f(x), g(y))$. The element $\alpha\times\beta$ of $\pi_{m+n}(X\times Y, X\vee Y, (x_0, y_0))$ represented by h depends only on α and β and is called a cross product of α and β. For m or $n=0$, $\alpha\times\beta$ has not been defined. But in this case, if we define $\alpha\times\beta$ by the conditions in I, § 2, (*a*) and (*b*), we have

Theorem 6. *The usual cross product operation satisfies the axiom of cross products described in* I, §2.

Proof.[2] We shall prove the theorem for the case $m>1$, $n>1$. The other cases can be treated by slight modifications. Let $f':(I^{m-1}, \dot{I}^{m-1})\to(\Omega X, x_0)$, $g':(I^{n-1}, \dot{I}^{n-1})\to(\Omega Y, y_0)$ be maps defined by $f'(x)(t)=f(x, t)$, $g'(y)(t)=g(y, t)$ for $x\in I^{m-1}$, $y\in I^{n-1}$, $t\in I$. Then they represent $\Omega\alpha\in\pi_{m-1}(\Omega X, x_0)$ and $\Omega\beta\in\pi_{n-1}(\Omega Y, y_0)$ respectively and $\Omega\alpha\times\Omega\beta\in\pi_{m+n-2}(\Omega X\times\Omega Y, \Omega X\vee\Omega Y, (x_0, y_0))$ is represented by a map $h':(I^{m-1}\times I^{n-1}, (I^{m-1}\times I^{n-1})^{\cdot})\to(\Omega X\times\Omega Y, \Omega X\vee\Omega Y)$ such that $h'(x, y)=(f'(x), g'(y))$. We shall show that

$$\partial\{h\}=(-1)^{n-1}\Omega^{-1}\varphi_\natural\{h'\}.$$

Define a map $\Phi:(I^{m-1}\times I^{n-1}\times I)^{\cdot}\to\Omega(X\vee Y)$ by

$$\Phi(x, y, t)=\begin{cases}\varphi h'(x, y), & (x, y)\in I^{m-1}\times I^{n-1},\ t=0,\\ H(h'(x, y), t), & (x, y)\in(I^{m-1}\times I^{n-1})^{\cdot},\ t\in I,\\ (x_0, y_0), & (x, y)\in I^{m-1}\times I^{n-1},\ t=1,\end{cases}$$

where H is a null-homotopy of $\varphi\mid\Omega X\vee\Omega Y$ defined in I, § 2. Then it can be shown by a straightforward calculation that Φ represents $\varphi_\natural\{h'\}$. Now define a map

$$\Phi':(I^{m-1}\times I\times I^{n-1}\times I)^{\cdot}\to X\vee Y \quad \text{by}$$

$$\Phi'(x, t, y, s)=\begin{cases}(x_0, y_0), & t\in\dot{I},\\ \Phi(x, y, s)(t), & \text{otherwise}.\end{cases}$$

Then we have that Φ' represents $(-1)^{n-1}\{\Phi\}=(-1)^{n-1}\varphi_\natural\{h'\}$. On the other hand, define a map $d:\dot{I}^2\to\dot{I}^2$ by

$$d(t, 0)=\begin{cases}(4t, 0), & 0\leqq t<\frac{1}{4},\\ (1, 4t-1), & \frac{1}{4}\leqq t<\frac{1}{2},\\ (3-4t, 1), & \frac{1}{2}\leqq t<\frac{3}{4},\\ (0, 4-4t), & \frac{3}{4}\leqq t\leqq 1,\end{cases}$$

$$d(1, s)=d(t, 1)=d(0, s)=(0, 0),$$

and extend d to I^2 linearly. Let $p_1, p_2: I\times I\to I$ be projections such that $p_1(t, s)=t$, $p_2(t, s)=s$. Then if we define a map $D:(I^{m-1}\times I\times I^{n-1}$

1) Cf. [1] or [4].

2) M. Tsuda pointed out that the same technique has been used by H. Samelson in the proof of his Theorem 1 in [3]; that is, if $\alpha\in\pi_m(X, x_0)$, $\beta\in\pi_n(X, x_0)$, $m>1$, $n>1$, then $h\Omega[\alpha, \beta]=(-1)^{m-1}(h\Omega\alpha * h\Omega\beta-(-1)^{(m-1)(n-1)}h\Omega\beta * h\Omega\alpha)$, where h denotes the Hurewicz homomorphism and $*$ the Pontryagin product in $H_*(\Omega X)$.

Added in proof. See also [5].

$\times I)^{\cdot} \to (I^{m-1} \times I \times I^{n-1} \times I)^{\cdot}$ by $D(x, t, y, s) = (x, p_1 d(t, s), y, p_2 d(t, s))$, D is a deformation. Let $\dot{h} = h \mid (I^m \times I^n)^{\cdot}$. Then we can prove that $\dot{h} \circ D \simeq \Phi'$ rel. 0 in $X \vee Y$ and this implies that

$$\Omega\partial(\alpha \times \beta) = (-1)^{n-1} \varphi_{\natural}(\Omega\alpha \times \Omega\beta).$$

The relation between the cross products and the Whitehead products in the usual theory is well known. Therefore the Whitehead product operation defined in I, § 2 coincides with the usual one in the usual homotopy theory.

References

[1] Blakers, A. L., and Massey, W. S.,: Products in homotopy theory, Ann. of Math., (2) **58**, 295–324 (1953).

[2] Hu, S. T.,: Axiomatic approach to the homotopy groups, Bull. A.M.S., **62**, 490–504 (1956).

[3] Samelson, H.,: A connection between the Whitehead and the Pontryagin product, Amer. J. Math., **76**, 744–752 (1953).

[4] Whitehead, G. W.,: A generalization of the Hopf invariant, Ann. of Math., (2) **51**, 192–237 (1950).

[5] Berstein, I., and Genea, T.,: Homotopy nilpotency, Illinois J. Math., **5**, 99–130 (1961).