47. On the Definition of the Cross and Whitehead Products in the Axiomatic Homotopy Theory. I

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1. Introduction. The axiomatic characterization of homotopy theory has been established by several authors [3–5]. In [3], S. T. Hu proposed the problem to present a new definition of the Whitehead products so that they might be fixed into the axiomatic scheme without appeal to the geometric representation of the homotopy groups. In this paper, we shall give an answer of this problem. First, the cross products will be defined; this is accomplished by using the commutator maps in loop spaces (cf. [2] and [6]). Then the Whitehead products will be given by the well-known formula $[\alpha, \beta] = F_{\#}\partial(\alpha\times\beta)$ for $\alpha\in\pi_m(X, x_0)$, $\beta\in\pi_n(X, x_0)$, where F: $X\vee X\to X$ is the folding map (cf. [1] or [7]).

2. The cross and the Whitehead products. Let (X, A, x_0) be a triplet of topological spaces and let $\Omega(X; A, x_0)$ be a path space of X consisting of maps σ: $I\to X$ such that $\sigma(0)\in A$, $\sigma(1)=x_0$. A constant path σ: $I\to X$ such that $\sigma(t)=x_0$ for all $t\in I$ will be denoted by the same letter x_0 and is considered as a base point of $\Omega(X; A, x_0)$. If any confusion does not occur, we denote $\Omega(X; A, x_0)$ simply by $\Omega(X, A)$ and $\Omega(X, x_0)$ by ΩX. For a map f: $(X, A, x_0)\to(Y, B, y_0)$, we shall denote by Ωf: $\Omega(X; A, x_0)\to\Omega(Y; B, y_0)$ a map such that $(\Omega f)(x)(t) = f(x(t))$.

Let $H=\{\pi, \#, \partial\}$ be a given axiomatic homotopy theory (we shall denote the multiplication in homotopy groups by $+$). It is shown in [3] that there exists an isomorphism

$$\Omega\colon \pi_n(X, A, x_0)\approx\pi_{n-1}(\Omega(X, A), x_0), \qquad \text{for } n>0.$$

Now, a cross product operation is a function defined for all pairs of homotopy groups $\pi_m(X, x_0)$ and $\pi_n(Y, y_0)$ of any two spaces (X, x_0) and (Y, y_0) and it is required to satisfy the following conditions.

Let $\alpha\in\pi_m(X, x_0)$, $\beta\in\pi_n(Y, y_0)$.

(a) In case $m=n=0$. According to the definition of 0-dimensional homotopy groups, α and β are considered as path-components of X and Y, respectively. Let $x\in\alpha$, $y\in\beta$. A path component γ of $X\times Y$ modulo $X\vee Y$ $(=X\times y_0\cup x_0\times Y)$ containing (x, y) depends only on α and β. A cross product of α and β is defined by

$$\alpha\times\beta=\gamma\in\pi_0(X\times Y, X\vee Y, (x_0, y_0)).$$

(b) In case $m=0$, $n>0$ or $m>0$, $n=0$. We may consider the case $m=0$, $n>0$; the other case is treated similarly. Let $x_1 \in \alpha$, where α is considered as a path-component of X modulo x_0. Define a map j_{x_1}; $(Y, y_0, y_0) \to (X \times Y, X \vee Y, (x_1, y_0))$ by $j_{x_1}(y)=(x_1, y)$ and define a cross product $\alpha \times \beta \in \pi_n(X \times Y, X \vee Y, (x_1, y_0))$ by

$$\alpha \times \beta = j_{x_1 \#}(\beta).$$

By using another point $\bar{x}_1 \in \alpha$, we can define $\alpha \mathbin{\bar{\times}} \beta \in \pi_n(X \times Y, X \vee Y, (\bar{x}_1, y_0))$. There exists a path σ: $I \to X$ such that $\sigma(0)=x_1$, $\sigma(1)=\bar{x}_1$. Define a path σ': $I \to X \times Y$ by $\sigma'(t)=(\sigma(t), y_0)$. Then σ' determines a transformation (homomorphism for $n>0$)

$$\sigma'_*\colon \pi_n(X \times Y, X \vee Y, (x_1, y_0)) \to \pi_n(X \times Y, X \vee Y, (\bar{x}_1, y_0))$$

and we can prove that $\alpha \mathbin{\bar{\times}} \beta = \sigma'_*(\alpha \times \beta)$ for any path σ. Now there exists an n-th abstract homotopy group of $X \times Y$ modulo $X \vee Y$ for a path component $\alpha \subset X$ (we may denote it by $\pi_n(X \times Y, X \vee Y, (\alpha, y_0))$ which is isomorphic to $\pi_n(X \times Y, X \vee Y, (x_1, y_0))$ for each $x_1 \in \alpha$ (cf. [3] §8). Hence we can consider $\alpha \times \beta$ as an element of $\pi_n(X \times Y, X \vee Y, (\alpha, y_0))$ and this is determined uniquely by α and β.

Remark. $j_{x_1 \#}$ is a monomorphism for the case $\alpha \not\ni x_0$. For if we define a map q_{x_1}: $(X \times Y, X \vee Y, (x_1, y_0)) \to (Y, y_0, y_0)$ by

$$q_{x_1}(x, y) = \begin{cases} y, & x \in \alpha, \\ y_0, & x \notin \alpha, \end{cases}$$

then $q_{x_1} \circ j_{x_1} = \text{identity}$.

(c) In case $m>0$, $n>0$, we require that

$$\alpha \times \beta \in \pi_{m+n}(X \times Y, X \vee Y, (x_0, y_0)).$$

Furthermore, cross products must satisfy the following axiom.

Axiom of cross products

$$\Omega\partial(\alpha \times \beta) = (-1)^{n-1} \varphi_\natural(\Omega\alpha \times \Omega\beta),$$

where φ is a commutator map of $\Omega X \times \Omega Y$ to $\Omega(X \vee Y)$ defined by

$$\varphi(x, y) = (x, y_0)(x_0, y)(x^{-1}, y_0)(x_0, y^{-1})$$

((x, y_0) is a loop in $X \vee Y$ such that $(x, y_0)(t)=(x(t), y_0)$ for $t \in I$) and $\varphi_\natural$ is a *modified induced transformation,* whose definition will be given in the following. First, we shall prove

Lemma 1. *Let $f:(X, x_0) \to (Y, y_0)$ be a map such that $f \mid A \simeq 0$ rel. x_0 for $A \subset X$ and let $H: A \times I \to Y$ be its homotopy such that $H(a, 0)=f(a)$, $H(a, 1)=y_0$, for $a \in A$. Then there exists a transformation (homomorphism for $n>0$)*

$$f_\natural : \pi_n(X, A, x_0) \to \pi_n(Y, y_0),$$

called a modified induced transformation by f and H.

Proof. It is clear for $n=0$. For $n>0$, define a map $\bar{f}$: $\Omega(X, A) \to \Omega(Y)$ by

$$\bar{f}(x)(t) = \begin{cases} H(x(0), 1-2t), & 0 \leqq t < \frac{1}{2}, \\ f(x(2t-1)), & \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

Then we have a required transformation $f_\natural$ by $f_\natural=\Omega^{-1}\bar{f}_{\#}\Omega$.

Now, if $m>1$, $n>1$ or $m=n=1$, $\varphi_\natural$ is defined by Lemma 1 by using the following nullhomotopy $H:(\Omega X\vee\Omega Y)\times I\to\Omega(X\vee Y)$ of $\varphi|\Omega X\vee\Omega Y$:

$$H(x, y_0, s)(t)=\begin{cases}(x(4t(1-s)), y_0), & 0\leqq t<\frac{1}{4},\\ (x(1-s), y_0), & \frac{1}{4}\leqq t<\frac{1}{2},\\ (x((3-4t)(1-s)), y_0), & \frac{1}{2}\leqq t<\frac{3}{4},\\ (x_0, y_0), & \frac{3}{4}\leqq t\leqq 1,\end{cases}$$

$$H(x_0, y, s)(t)=\begin{cases}(x_0, y_0), & 0\leqq t<\frac{1}{4},\\ (x_0, y((4t-1)(1-s))), & \frac{1}{4}\leqq t<\frac{1}{2},\\ (x_0, y(1-s)), & \frac{1}{2}\leqq t<\frac{3}{4},\\ (x_0, y((4-4t)(1-s))), & \frac{3}{4}\leqq t\leqq 1,\end{cases}$$

for $x\in\Omega X$, $y\in\Omega Y$, $s\in I$.

If $m=1$, $n>1$, φ determines a transformation

$\varphi_\natural:\pi_{n-1}(\Omega X\times\Omega Y, \Omega X\vee\Omega Y, (x_1, y_0))\to\pi_{n-1}(\Omega(X\vee Y), (x_1x_0x_1^{-1}x_0^{-1}, y_0))$, where x_1 is a point in a path-component $\Omega\alpha\subset\Omega X$. Since a map $\sigma: I\to\Omega(X\vee Y)$ such that $\sigma(s)(t)=H(x_1, y_0, s)(t)$ gives a path connecting $(x_1x_0x_1^{-1}x_0^{-1}, y_0)$ with (x_0, y_0), there exists a transformation

$$\sigma_*:\pi_{n-1}(\Omega(X\vee Y), (x_1x_0x_1^{-1}x_0^{-1}, y_0))\to\pi_{n-1}(\Omega(X\vee Y), (x_0, y_0)).$$

we can show that the element $(\sigma_*\circ\varphi_\natural)(\Omega\alpha\times\Omega\beta)\in\pi_{n-1}(\Omega(X\vee Y), (x_0, y_0))$ does not depend on the choice of a point $x_1\in\Omega\alpha$. Then, using the same letter, we define a transformation

$\varphi_\natural:\pi_{n-1}(\Omega X\times\Omega X, \Omega X\vee\Omega Y, (x_1, y_0))\to\pi_{n-1}(\Omega(X\vee Y), (x_0, y_0))$, by taking $\varphi_\natural=\sigma_*\circ\varphi_\natural$ for $m=1$, $n>1$. It is similarly defined for $m>1$, $n=1$.

Now we proceed to define the Whitehead products in the axiomatic homotopy theory.

Let $F: X\vee X\to X$ be the folding map, i.e. $F(x, x_0)=x$, $F(x_0, x)=x$ for $x\in X$. A Whitehead product $[\alpha, \beta]\in\pi_{m+n-1}(X, x_0)$ of $\alpha\in\pi_m(X, x_0)$ and $\beta\in\pi_n(X, x_0)$, $m>0$, $n>0$ is defined by the formula

$$[\alpha, \beta]=F_{\#}\partial(\alpha\times\beta).$$

Remark 1. There are some possibilities in the determination of the sign in the axiom of cross products. For example, we can take $(-1)^m$ instead of $(-1)^{n-1}$; if we take $\alpha\boxtimes\beta$ by $\alpha\boxtimes\beta=(-1)^{mn}\alpha\times\beta$, then $\Omega\partial(\alpha\boxtimes\beta)=(-1)^m\ \varphi_\natural(\Omega\alpha\boxtimes\Omega\beta)$ holds.

Remark 2. In case $\alpha\in\pi_m(X, A)$ and $\beta\in\pi_n(Y, B)$, their cross product $\alpha\times\beta\in\pi_{m+n}(X\times Y, A\times Y\cup X\times B)$ can be defined also by the above conditions (we shall need to modify slightly the definition of φ).

In a subsequent paper, we shall prove the existence and uniqueness of the cross and the Whitehead products.

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