45. A Note on Hausdorff Spaces with the Star-finite Property. II

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K. Morita [4] constructed, for every metric space R, a 0-dimensional metric space S and a closed continuous mapping f of S onto R such that $f^{-1}(x)$ is compact for every point x of R. The purpose of this note is to give an analogous proposition to this theorem for the case when R is paracompact Hausdorff. As for the terminologies and the notations used in this note, refer to my previous note [7].

Theorem 1. *Let f be a closed continuous mapping of a regular space R onto a topological space S with the star-finite property such that $f^{-1}(y)$ has the Lindelöf property for every point y of S. Then R has the star-finite property.*

Proof. Let $\mathfrak{U}$ be an arbitrary open covering of R. For every point y of S let $\mathfrak{U}_y=\{U_\alpha;\ \alpha\in A_y\}$ be a subcollection of $\mathfrak{U}$ which consists of countable elements such that $\mathfrak{U}_y$ covers $f^{-1}(y)$. Let $U_y=\cup\{U_\alpha;\ \alpha\in A_y\}$ and $V_y=S-f(R-U_y)$. Then V_y is an open neighborhood of y. Let $\mathfrak{B}=\{V_\beta;\ \beta\in B\}$ be a star-finite open covering of S which refines $\{V_y;\ y\in S\}$. Let us define a (single-valued) mapping φ of B into S such that $\varphi(\beta)=y$ yields $V_\beta\subset V_y$. Let $W_y=f^{-1}(V_y)$ and $W_\beta=f^{-1}(V_\beta)$. Then we can prove that $\mathfrak{W}=\{W_\beta\cap U_\alpha;\ \alpha\in A_{\varphi(\beta)},\ \beta\in B\}$ is a star-countable open covering of R.

To show that $\mathfrak{W}$ covers R, let x be an arbitrary point of R. Then there exists $\beta\in B$ such that $x\in W_\beta$. Since $V_\beta\subset V_{\varphi(\beta)}$, we get $W_\beta\subset W_{\varphi(\beta)}$. Since $W_{\varphi(\beta)}\subset U_{\varphi(\beta)}$ and $U_{\varphi(\beta)}=\cup\{U_\alpha;\ \alpha\in A_{\varphi(\beta)}\}$, there exists an $\alpha\in A_{\varphi(\beta)}$ such that $x\in U_\alpha$. Hence $\mathfrak{W}$ is an open covering of R. On the other hand the star-countability of $\mathfrak{W}$ is almost evident. Therefore we can conclude that R has the star-countable property. Since in general a regular space with the star-countable property has the star-finite property by Yu. Smirnov [9],[1] R has so and the theorem is proved.

Theorem 2. *Let R be a non-empty paracompact Hausdorff space. Then there exist a paracompact Hausdorff space A with* dim $A=0$ *and a closed continuous mapping f of A onto R such that $f^{-1}(x)$ is compact for every point x of R.*

Proof. Let $\{\mathfrak{F}_\lambda=\{F_\alpha;\ \alpha\in A_\lambda\};\ \lambda\in\Lambda\}$ be the collection of all locally finite colsed coverings of R. Let A be the aggregate of points a

1) This theorem is also almost essentially proved in Morita [5].

$=(\alpha_\lambda;\ \lambda\in\Lambda)$ of the product space $\Pi\{A_\lambda;\ \lambda\in\Lambda\}$, where A_λ are topological spaces with the discrete topology, such that $\frown\{F_{\alpha_\lambda};\ \lambda\in\Lambda\}\neq\phi$. When $\frown\{F_{\alpha_\lambda};\ \lambda\in\Lambda\}$ is not empty, it is a single point. Define $f: A\to R$ as $f(a)=\frown\{F_{\pi_\lambda(a)};\ \lambda\in\Lambda\}$, where $\pi_\lambda: B\to A_\lambda$, $\lambda\in\Lambda$, is the restriction of the projection defined on ΠA_λ into A_λ. It can easily be seen that f is continuous and onto.

To show the closedness of f, let B be an arbitrary non-empty closed subset of A and x an arbitrary point of $\overline{f(B)}$. Let λ be an arbitrary element of Λ. Let $B_\lambda=\{\alpha;\ x\in F_\alpha\in\mathfrak{F}_\lambda\}$; then $U_\lambda=R-\smile\{F_\alpha;\ \alpha\in A_\lambda-B_\lambda\}$ is an open neighborhood of x by the local finiteness of $\mathfrak{F}_\lambda$. Since $f(B)\frown U_\lambda\neq\phi$, it holds that $B\frown f^{-1}(U_\lambda)\neq\phi$. Since $f^{-1}(U_\lambda)\subset\smile\{\pi_\lambda^{-1}(\alpha);\ \alpha\in B_\lambda\}$, there exists an index $\alpha(\lambda)\in B_\lambda$ with $\pi_\lambda^{-1}(\alpha(\lambda))\frown B\neq\phi$.

Let $a=(\alpha(\lambda);\ \lambda\in\Lambda)$; then it is easy to see that $f(a)=x$. Since, for any λ, $\pi_\lambda^{-1}(\pi_\lambda(a))\frown B=\pi_\lambda^{-1}(\alpha(\lambda))\frown B\neq\phi$, a is a point of $\overline{B}=B$. Therefore we get $x=f(a)\in f(B)$ and hence $\overline{f(B)}\subset f(B)$. Thus the closedness of f is proved. Moreover $f^{-1}(x)$ is compact, since $f^{-1}(x)=\Pi\{B_\lambda;\ \lambda\in\Lambda\}$ and B_λ is finite for every $\lambda\in\Lambda$.

Finally let us prove that A is a paracompact Hausdorff space with $\dim A=0$. Let $\mathfrak{U}$ be an arbitrary open covering of A; then $\mathfrak{U}$ can be refined by a covering $\mathfrak{V}$ whose elements are open and closed, by the equality $\operatorname{ind} A=0$. Since, for any $x\in R$, $f^{-1}(x)$ is compact, there exist a finite number of elements $V_{x,1},\cdots,V_{x,m(x)}$ of $\mathfrak{V}$ with $f^{-1}(x)\subset V_{x,1}\smile\cdots\smile V_{x,m(x)}=W_x$, where we can put $V_{x,1}=\phi$, $x\in R$, without loss of generality. Put $D(x)=R-f(A-W_x)$; then there exists an index $\lambda_0\in\Lambda$ such that $\mathfrak{F}_{\lambda_0}$ refines $\{D(x);\ x\in R\}$. Since i) $\{\pi_{\lambda_0}^{-1}(\alpha);\ \alpha\in A_\lambda\}$ refines $\{f^{-1}(D(x));\ x\in R\}$ and the latter refines $\{W_x;\ x\in R\}$ and ii) the order of $\{\pi_{\lambda_0}^{-1}(\alpha);\ \alpha\in A_\lambda\}$ is 1, we can prove, by an easy transfinite induction on $x\in R$, the existence of an open covering $\{U_x;\ x\in R\}$ of order 1 with $U_x\subset W_x$ for every $x\in R$.

Let $\mathfrak{E}=\{U_x\frown(V_{x,i}-\smile_{j<i}V_{x,j});\ i=2,\cdots,m(x),\ x\in R\}$; then $\mathfrak{E}$ is an open covering of A of order 1 which refines $\mathfrak{U}$. Thus A is a paracompact Hausdorff space with $\dim A=0$ and the theorem is proved.

Remark. An analogous result to our Theorem 2 has been obtained independently by V. Ponomarev [8]. He proves that for any normal space R there exist a completely regular space A with $\operatorname{ind} A=0$ and a closed continuous mapping f of A onto R such that i) $f^{-1}(x)$ is compact for every x of R, ii) $f(A_1)\neq R$ for any proper closed subset A_1 of A,[2] iii) $\tau A=\tau R$, where τA and τR denote respectively the topological weights[3] of A and R. We shall show in the following that this theorem is valid even if R is completely regular. He says

2) A mapping with this property ii) is called *irreducible*.

3) The *topological weight* of a topological space is the minimum of the cardinal numbers of its open bases.

also that A cited in his theorem is normal. But it seems that, as far as I know, there has been no paper which assures the normality of A. I hope that he will make a public expression of his proof.

Lemma 1. *Let R be a topological space, S a space and f a mapping of R onto S such that $f^{-1}(y)$ is compact for every point $y \in S$. Then there exists a closed subset R_1 of R such that $f \mid R_1$ is irreducible.*

Proof. Let $\mathfrak{F}=\{F_\alpha;\ \alpha \in A\}$ be the family of all closed subsets F_α of R such that $f(F_\alpha)=S$. Let us introduce into $\mathfrak{F}$ the semi-order $<$ such that $F_\alpha < F_\beta$ if and only if $F_\alpha \supset F_\beta$. Let $\mathfrak{F}_1=\{F_\alpha;\ \alpha \in A_1\}$ be an arbitrary linearly ordered subset of $\mathfrak{F}$ and y an arbitrary point of S. Then $\{F_\alpha \frown f^{-1}(y);\ \alpha \in A_1\}$ has clearly the finite intersection property. Hence $\frown\{F_\alpha;\ \alpha \in A_1\} \frown f^{-1}(y) \neq \phi$, which proves $\frown\{F_\alpha;\ \alpha \in A_1\} \in \mathfrak{F}$. Thus $\mathfrak{F}_1$ has an upper bound in $\mathfrak{F}$. Therefore by Zorn's lemma $\mathfrak{F}$ has a maximal element R_1. $f \mid R_1$ is evidently irreducible.

Theorem 3. *Let R be a non-empty completely regular space. Then there exist a completely regular space A and a closed continuous mapping f of A onto R which satisfy the following conditions.*
(1) *$f^{-1}(x)$ is compact for every point $x \in R$.*
(2) *f is irreducible.*
(3) $\operatorname{ind} A=0$.
(4) $\tau A \leq \tau R$.

Proof. Embed R densely into a compact Hausdorff space S with $\tau R=\tau S$; this is possible. Let $\mathfrak{U}=\{U_\xi;\ \xi \in \Xi\}$ be an open basis of S with $|\Xi|=\tau R$. Let $\mathfrak{M}=\{M_\sigma;\ \sigma \in \Sigma_1\}$ be the family of all finite subsets M_σ of Ξ; then $|\mathfrak{M}|=|\Xi|=\tau R$. Hence we have $|\mathrm{F}|=\tau R$, where $\mathrm{F}=\{\mathfrak{F}_\sigma;\ \sigma \in \Sigma\}=\{\mathfrak{F}_\sigma=\{\overline{U}_\xi;\ \xi \in M_\sigma\};\ M_\sigma \in \mathfrak{M},\ \smile\{\overline{U}_\xi;\ \xi \in M_\sigma\}=S\}$. Consider the product space $\Pi\{M_\sigma;\ \sigma \in \Sigma\}$, where M_σ are topological spaces with the discrete topology. Then $\tau\Pi\{M_\sigma;\ \sigma \in \Sigma\} \leq |\mathfrak{M}|=\tau R$. Let B be the aggregate of points $a=(\xi(\sigma);\ \sigma \in \Sigma)$ of ΠM_σ such that $\frown \overline{U}_{\xi(\sigma)} \neq \phi$. Then $\tau B \leq \tau \Pi M_\sigma \leq \tau R$. When $\frown\{\overline{U}_{\xi(\sigma)};\ \sigma \in \Sigma\}$ is not empty, it consists of a single point. Define $g: B \to S$ as $g(a)=\frown\{\overline{U}_{\pi_\sigma(a)};\ \sigma \in \Sigma\}$. Then by the same argument used in the proof of Theorem 2 we can know that i) B is a compact Hausdorff space with $\dim B=0$, ii) g is continuous and onto.

Let $A_1=g^{-1}(R)$ and $g_1=g \mid A_1$. Then the following conditions are satisfied: i) g_1 is closed continuous and onto. ii) For every point $x \in R$, $g_1^{-1}(x)$ is compact. iii) $\tau A_1 \leq \tau B \leq \tau R$. iv) $\operatorname{ind} A_1=0$. By Lemma 1 there exists a closed subset A of A_1 such that $f=g_1 \mid A$ is irreducible. A and f thus obtained satisfy all the conditions required and the theorem is proved.

Lemma 2. *Let f be a closed continuous mapping of a topological space R onto a paracompact space S such that $f^{-1}(y)$ is compact for every point $y \in S$. Then R is paracompact.*

Cf. S. Hanai [2] or M. Henriksen-R. Isbell [3, Theorem 2.2].

Corollary. *Let R be a non-empty paracompact Hausdorff*[4] S_σ-*space.*[5] *Then there exist a paracompact Hausdorff* S_σ-*space A with* $\dim A=0$ *and a closed continuous mapping f of A onto R which satisfy the following conditions.*

(1) *$f^{-1}(x)$ is compact for every point x of R.*

(2) *f is irreducible.*

(3) $\dim A=0$.

(4) *$\tau A \leq \tau R$.*

Proof. By Theorem 2 there exist a completely regular space A with $\operatorname{ind} A=0$ and a closed continuous mapping f of A onto R which satisfy the conditions (1), (2), (4). Let $R=\bigcup_{i=1}^{\infty} R_i$ where R_i, $i=1, 2, \cdots$, are non-empty closed subsets with the star-finite property. Then $A_i=f^{-1}(R_i)$, $i=1, 2, \cdots$, is a closed subset of A with the star-finite property by Theorem 1. Hence by Morita [6, Theorem 5.2] we get $\dim A_i=0$. Moreover by Lemma 2 A is paracompact and hence A is normal by J. Dieudonné [1]. Therefore by the sum theorem we get $\dim A=0$ and the corollary is proved.

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4) This condition of R can be replaced with a weaker condition, collectionwise normality of R, since the following proposition is as can easily be seen valid: Let F_i, $i=1,2,\cdots$, be pointwise paracompact closed subsets of a collectionwise normal space; then $\bigcup F_i$ is paracompact.

5) A space which is the sum of a countable number of closed subsets with the star-finite property is called an S_σ-*space*. This notion is due to Morita.