44. On Metric General Connections

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In this note, the author will show that the Levi-Civita's connections of Riemann spaces can be generalized in the theory of general connections under some conditions on an n-dimensional differentiable manifold $\mathfrak{X}$. He will use the notations in [3].

1. A tensor P of type (1, 1) is called *normal* when P as a homomorphism of the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$ is an isomorphism on each $P(T_x(\mathfrak{X}))=P_x(\mathfrak{X})$, $x\in\mathfrak{X}$, and $\dim P_x(\mathfrak{X})$ is constant. Let us assume that P is normal and put $\dim P_x(\mathfrak{X})=m$. If we put $N_x(\mathfrak{X})=$the kernel of P on $T_x(\mathfrak{X})$, then we have

$$T_x(\mathfrak{X})=P_x(\mathfrak{X})+N_x(\mathfrak{X}).$$

According to the direct sum decomposition of $T(\mathfrak{X})$, we define two projections A and N which map $T_x(\mathfrak{X})$ onto $P_x(\mathfrak{X})$ and $N_x(\mathfrak{X})$ respectively at each point x of $\mathfrak{X}$. A and N may be considered as tensors of type (1, 1) of $\mathfrak{X}$. Clearly we have $A+N=I$, $A^2=A$, $N^2=N$, $AN=NA=0$, $AP=PA=P$ and $NP=PN=0$, where I denotes the fundamental unit tensor of type (1, 1).

Now, we say that a normal tensor P is *orthogonally related with* a non-singular symmetric tensor $G=g_{ij}du^i\otimes du^j$, if $P_x(\mathfrak{X})$ and $N_x(\mathfrak{X})$ are mutually orthogonal with respect to G, regarding G as a metric tensor.

A general connection Γ, which is locally written as

$$\Gamma=\partial u_i\otimes(P^i_jd^2u^j+\Gamma^i_{jh}du^j\otimes du^h),^{1)}\quad \partial u_i=\partial/\partial u^i,$$

is called *normal*, if the tensor $P=\lambda(\Gamma)$[2] $=\partial u_i\otimes P^i_jdu^j$ is normal.

A normal general connection Γ is called *proper*,[3] if the tensor of type (1, 2) with local components $N^i_k\Gamma^k_{jh}$ vanishes, where N^i_j are the local components of the tensor N.

We say that a general connection Γ satisfies *the metric condition* for a symmetric covariant tensor $G=g_{ij}du^i\otimes du^j$, if

$$(1)\qquad DG=g_{ij,h}du^i\otimes du^j\otimes du^h=0,$$

where DG denotes the covariant differential of G with respect to Γ.[4] On the metric condition, the following theorem holds good as in the

1) See [3].

2) See [3], §2.

3) On the geometrical meaning of this condition, see Theorem 5.2 of [4]. In general, Γ^i_{jh} are not local components of a tensor of type (1, 3) as the classical affine connections but $N^i_k\Gamma^k_{jh}$ are so.

4) See (2.15) of [3].

classical case.

Theorem 1. *Let Γ be a metric general connection with respect to a symmetric covariant tensor G of order 2. For any two contravariant vectors with local components v^i and w^i defined on a curve $u^i=u^i(t)$ along which they are covariantly constant, the scalar $g_{hk}P^h_iP^k_jv^iw^j$ is constant.*[5] *Conversely, if Γ has the property for any curve, then Γ is metric with respect to G.*

Proof. The metric condition (1) is written as

$$g_{ij,h}=\frac{\partial g_{lk}}{\partial u^h}P^l_iP^k_j-g_{lk}\Lambda^l_{ih}P^k_j-g_{lk}P^l_i\Lambda^k_{jh}=0,$$

where

$$\Lambda^i_{jh}=\Gamma^i_{jh}-\frac{\partial P^i_j}{\partial u^h}.$$

Clearly, $g_{ij,h}$ can be also written as

$$g_{ij,h}=\frac{\partial}{\partial u^h}(g_{lk}P^l_iP^k_j)-g_{lk}\Gamma^l_{ih}P^k_j-g_{lk}P^l_i\Gamma^k_{jh}. \tag{2}$$

Now, let $V=v^i\partial u_i$, $W=w^i\partial u_i$ be covariantly constant along a curve $u^i=u^i(t)$, then it must hold good

$$Dv^i=P^i_jdv^j+\Gamma^i_{jh}v^jdu^h=0,\quad Dw^i=P^i_jdw^j+\Gamma^i_{jh}w^jdu^h=0 \tag{3}$$

along the curve. Hence, we have

$$\begin{aligned}d(g_{lk}P^l_iP^k_jv^iw^j)&=d(g_{lk}P^l_iP^k_j)v^iw^j+g_{lk}(P^l_idv^i)P^k_jw^j+g_{lk}P^l_iv^i(P^k_jdw^j)\\&=[d(g_{lk}P^l_iP^k_j)-g_{lk}\Gamma^l_{ih}P^k_jdu^h-g_{lk}P^l_i\Gamma^k_{jh}du^h]v^iw^j\\&=g_{ij,h}v^iw^jdu^h.\end{aligned}$$

Since, at any point of a curve, we have solutions of (3) with any initial values at the point, the condition: $g_{ij,h}=0$ is equivalent to the condition: $d(g_{lk}P^l_iP^k_jv^iw^j)=0$ for any curve and any two contravariant vector fields v^i and w^i covariantly constant along the curve. q.e.d.

2. Now we shall prove the following

Theorem 2. *Let $P=P^i_j\partial u_i\otimes du^j$ and $G=g_{ij}du^i\otimes du^j$ be a normal tensor and a non-singular symmetric tensor on $\mathfrak{X}$ such that P is orthogonally related with G. Then, there exist normal general connections Γ which satisfy the following conditions:*

(i) *$P=\lambda(\Gamma)$,* (ii) *Γ is proper, and*

(iii) *Γ is metric with respect to G.*

Furthermore, if we add the condition:

(iv) $$S^i_{kh}A^k_j=\frac{1}{2}A^i_l(P^l_{k;h}-P^l_{h;k})A^k_j,$$

where A^i_j are the local components of A, $S^i_{jh}=\frac{1}{2}(\Gamma^i_{jh}-\Gamma^i_{hj})$ and the semi-colon ";" denotes the covariant derivatives with respect to the Levi-Civita's connection made by G, then Γ is uniquely determined.

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5) For general connections, the covariant differentiation and the contraction are not necessarily commutative. See [3], §2.

The condition (iv) is a generalization of the symmetric condition in the classical case, because we have $A^i_j=\delta^i_j$ and $P^i_{j;h}=0$, when $P^i_j=\delta^i_j$.

Proof. Now, let be given two tensors P and G as stated in the theorem and assume that there exists a normal general connection Γ satisfying the conditions (i), (ii) and (iii).

If we put

$$\bar{g}_{ij}=g_{kh}P^k_iP^h_j,\ \bar{\Gamma}^i_{jh}=\frac{1}{2}(\Gamma^i_{jh}+\Gamma^i_{hj}),\ S^i_{jh}=\frac{1}{2}(\Gamma^i_{jh}-\Gamma^i_{hj}),$$

the condition (iii) can be written as

$$\frac{\partial\bar{g}_{ij}}{\partial u^h}=\bar{\Gamma}_{ikh}P^k_j+\bar{\Gamma}_{jkh}P^k_i+S_{ikh}P^k_j+S_{jkh}P^k_i, \tag{4}$$

where

$$\bar{\Gamma}_{ikh}=g_{kl}\bar{\Gamma}^l_{ih},\quad S_{ikh}=g_{kl}S^l_{ih}.$$

As easily seen, S^i_{jh} are the local components of a tensor of type (1, 2) as in the classical case. If we denote the Christoffel symbols of the first kind made by $\bar{g}_{ij}$ by

$$\overline{[ij,h]}=\frac{1}{2}\left(\frac{\partial\bar{g}_{ih}}{\partial u^j}+\frac{\partial\bar{g}_{hj}}{\partial u^i}-\frac{\partial\bar{g}_{ij}}{\partial u^h}\right),$$

then (4) is clearly equivalent to

$$\overline{[ij,h]}=\bar{\Gamma}_{ikj}P^k_h+S_{hki}P^k_j+S_{hkj}P^k_i. \tag{5}$$

Now, let Q be the homomorphism of $T(\mathfrak{X})$ which operates as $Q=P^{-1}$ on each $P_x(\mathfrak{X})$ and $Q=0$ on each $N_x(\mathfrak{X})$. Then we have easily

$$PQ=QP=A,\quad QN=NQ=0.$$

Let Q^i_j be the local components of Q. Then, we get from (5)

$$\overline{[ij,l]}Q^l_h=\bar{\Gamma}_{ilj}A^l_h+(S_{lki}P^k_j+S_{lkj}P^k_i)Q^l_h \tag{6}$$

and

$$\overline{[ij,l]}N^l_h=(S_{lki}P^k_j+S_{lkj}P^k_i)N^l_h. \tag{7}$$

Making use of the relations between A, N, P and Q, we can easily see that (5) is derived from (6) and (7).

The condition (ii) can be written as

$$N^i_k\bar{\Gamma}^k_{jh}=0 \tag{8}$$

and

$$N^i_kS^k_{jh}=0. \tag{9}$$

Therefore, in order to obtain a normal general connection Γ satisfying the conditions (i), (ii) and (iii), it is sufficient that we solve firstly the equations (7) and (9) with respect to $S^i_{jh}=-S^i_{hj}$, and secondly the equations (6) and (8) with respect to $\bar{\Gamma}^i_{jh}=\bar{\Gamma}^i_{hj}$, using the solution S^i_{jh} of (7) and (9).

In the first place, we shall show that there exists a solution of (7) and (9) under the condition between P and G. We have

$$2\overline{[ij,l]}N^l_h=\left\{\frac{\partial}{\partial u^j}(g_{st}P^s_iP^t_l)+\frac{\partial}{\partial u^i}(g_{st}P^s_jP^t_l)-\frac{\partial}{\partial u^l}(g_{st}P^s_iP^t_j)\right\}N^l_h$$

$$=\Big\{g_{kt}\Big(\frac{\partial P^t_l}{\partial u^i}-\frac{\partial P^t_i}{\partial u^l}\Big)P^k_j+g_{kt}\Big(\frac{\partial P^t_l}{\partial u^j}-\frac{\partial P^t_j}{\partial u^l}\Big)P^k_i$$
$$-([sl,t]+[tl,s])P^s_iP^t_j\Big\}N^l_h,$$

that is

(10) $$[\overline{ij,l}]N^l_h=\Big\{\frac{1}{2}g_{kt}(P^t_{l;i}-P^t_{i;l})P^k_j+\frac{1}{2}g_{kt}(P^t_{l;j}-P^t_{j;l})P^k_i\Big\}N^l_h,$$

where $[ij,h]$ are the Christoffel symbols of the first kind made by g_{ij}.

Comparing (10) with (7), we define a tensor of type (1, 3) with local components

(11) $$\bar{S}^i_{jh}=\frac{1}{2}A^i_k(P^k_{j;h}-P^k_{h;j}),$$

then we have

$$(\bar{S}_{lki}P^k_j+\bar{S}_{lkj}P^k_i)N^l_h=g_{kt}(\bar{S}^t_{li}P^k_j+\bar{S}^t_{lj}P^k_i)N^l_h$$
$$=\Big\{\frac{1}{2}g_{kt}(P^t_{l;i}-P^t_{i;l})P^k_j+\frac{1}{2}g_{kt}(P^t_{l;j}-P^t_{j;l})P^k_i\Big\}N^l_h$$
$$-\frac{1}{2}\Big\{g_{ks}N^s_tP^k_j(P^t_{l;i}-P^t_{i;l})+g_{ks}N^s_tP^k_i(P^t_{l;j}-P^t_{j;l})\Big\}N^l_h$$
$$=[\overline{ij,l}]N^l_h,$$

since we have $g_{ij}P^i_kN^j_h=0$. On the other hand, we have

$$N^i_k\bar{S}^k_{jh}=\frac{1}{2}N^i_kA^k_l(P^l_{j;h}-P^l_{h;j})=0.$$

Thus, we have proved that the tensor $\bar{S}^i_{jh}$ is a solution of (7) and (9).

Now, if we put

$$S^i_{jh}-\bar{S}^i_{jh}=X^i_{jh}=-X^i_{hj},$$

they must satisfy the equations

(12) $$(X_{lki}P^k_j+X_{lkj}P^k_i)N^l_h=0,$$

(13) $$N^i_kX^k_{jh}=0.$$

Furthermore, supposing the condition (iv), it can be written as

(14) $$X^i_{kh}A^k_j=0.$$

It is equivalent to

$$X^i_{kh}N^k_j=X^i_{jh},$$

and so (12) can be written as

$$X_{hki}P^k_j+X_{hkj}P^k_i=0.$$

Hence $Y_{ihj}=X_{ikj}P^k_h$ are skew-symmetric with respect to the indices i, h, j. Using (14), we get

$$Y_{ikj}A^k_h=X_{ilj}P^l_kA^k_h=X_{ilj}P^l_h=Y_{ihj}$$
$$=-Y_{kij}A^k_h=-X_{klj}A^k_hP^l_i=0,$$

hence

(15) $$X_{ikj}A^k_h=0.$$

On the other hand, from the assumption that P is orthogonally related with G, we have

(16) $$g_{ij}A^i_kN^j_h=0.$$

Using these relations (15), (13) and (16), we have

$$X_{ihj}=X_{ikj}N^k_h=X^l_{ij}g_{kl}N^k_h=X^t_{ij}g_{kl}A^l_tN^k_h=0.$$

Thus, we have proved that under the conditions (i)–(iv), there exists a unique solution S^i_{jh} which is the skew-symmetric part of Γ^i_{jh}.

In the next place, we shall show that there exists a unique solution $\overline{\Gamma}^i_{jh}$ of (6) and (8) under the conditions (i), (ii), and (iii) regarding S^i_{jh} as a known tensor.

Let us take a local field of frame $\{V_\lambda\}$ of the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$ such that $\{V_1,\cdots,V_m\}$ and $\{V_{m+1},\cdots,V_n\}$ are frames of $P_x(\mathfrak{X})$ and $N_x(\mathfrak{X})$ at each point x respectively. Let $\{U^\lambda\}$ be the dual frame of $\{V_\lambda\}$. Then we have $A^i_j=V^i_\alpha U^\alpha_j$, $N^i_j=V^i_B U^B_j$.[6] $U^{\lambda i}=g^{ij}U^\lambda_j$ can be written as $U^{\lambda i}=C^{\lambda\mu}V^i_\mu$, hence we have

$$C^{\lambda\mu}=g^{ij}U^\lambda_iU^\mu_j=C^{\mu\lambda},\quad |C^{\lambda\mu}|\neq 0.$$

If we put

$$C_{\lambda\mu}=g_{ij}V^i_\lambda V^j_\mu=C_{\mu\lambda},$$

the matrix $(C_{\lambda\mu})$ is the inverse of the matrix $(C^{\lambda\mu})$. By virtue of the assumption of this theorem, we have

$$C_{\alpha A}=C^{\beta B}=0.$$

Now, (8) is clearly equivalent to

$$U^A_k\overline{\Gamma}^k_{ij}=0,\quad A=m+1,\cdots,n,$$

and so we have

$$\overline{\Gamma}_{ilj}U^{Al}=\overline{\Gamma}_{ilj}C^{AB}V^l_B=0,$$

hence

$$\overline{\Gamma}_{ilj}N^l_h=0. \tag{17}$$

From (6) and (17), we get

$$\overline{\Gamma}_{ihj}=([\overline{ij,l}]-S_{lki}P^k_j-S_{lkj}P^k_i)Q^l_h$$

and

$$\overline{\Gamma}^h_{ij}=([\overline{ij,l}]-S_{lki}P^k_j-S_{lkj}P^k_i)Q^l_pg^{ph}. \tag{18}$$

Conversely, $\overline{\Gamma}^h_{ij}$ given by (18) satisfy (6) and (8), as is easily seen.

Lastly, we must show that $\partial u_i\otimes(P^i_jd^2u^j+(\overline{\Gamma}^i_{jh}+S^i_{jh})du^j\otimes du^h)$ is a general connection. It is sufficient to show that $\partial u_i\otimes(P^i_jd^2u^j+\overline{\Gamma}^i_{jh}du^j\otimes du^h)$ is a general connection. Here, let us denote the components in another coordinate system v^λ by the notations with stars. Then we have

$$\overline{\Gamma}^h_{ij}=\left(\overline{g}^*_{\lambda\mu}\frac{\partial^2v^\lambda}{\partial u^j\partial u^i}\frac{\partial v^\mu}{\partial u^l}+([\overline{\lambda\mu,\rho}]^*-S^*_{\rho\tau\lambda}P^{*\tau}_{\ \mu}-S^*_{\rho\tau\mu}P^{*\tau}_{\ \lambda})\frac{\partial v^\lambda}{\partial u^i}\frac{\partial v^\mu}{\partial u^j}\frac{\partial v^\rho}{\partial u^l}\right)Q^l_kg^{kh}$$

$$=\frac{\partial u^h}{\partial v^\nu}\left(g^{*\nu\rho}Q^{*\mu}_{\ \rho}\overline{g}^*_{\mu\lambda}\frac{\partial^2v^\lambda}{\partial u^j\partial u^i}+\overline{\Gamma}^{*\nu}_{\lambda\mu}\frac{\partial v^\lambda}{\partial u^i}\frac{\partial v^\mu}{\partial u^j}\right).$$

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6) Indices run as follows: $\lambda,\mu,\nu,\cdots=1,2,\cdots,n$; $\alpha,\beta,\cdots=1,2,\cdots,m$; $A,B,\cdots=m+1,\cdots,n$.

Since we have

$$g^{ik}Q_k^h\bar{g}_{hj}=g^{ik}Q_k^h g_{lt}P_h^l P_j^t=g^{ik}A_k^l g_{lh}P_j^h$$
$$=P_j^i-g^{ik}g_{lh}N_k^l P_j^h=P_j^i,$$

the above equation can be written as

$$\bar{\Gamma}_{ij}^h=\frac{\partial u^h}{\partial v^\nu}\left(P^{*\nu}_\lambda\frac{\partial^2 v^\lambda}{\partial u^j\partial u^i}+\bar{\Gamma}^{*\nu}_{\lambda\mu}\frac{\partial v^\lambda}{\partial u^i}\frac{\partial v^\mu}{\partial u^j}\right).$$

This shows that $\partial u_i\otimes(P_j^i d^2u^j+\bar{\Gamma}_{jh}^i du^j\otimes du^h)$ determines a general connection. Thus, we have proved the theorem. q.e.d.

References

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