43. Continuity Properties on the Retardation in the Theory of Difference-Differential Equations

By Shohei SUGIYAMA

Department of Mathematics, School of Science and Engineering, Waseda University, Tokyo

(Comm. by Z. SUETUNA, M.J.A., April 12, 1961)

Introduction. In their paper,*) Bellman and Cooke have discussed the uniform convergence of solutions of a certain type of difference-differential equations as the retardation approaches zero, for which they applied the successive approximation method as a tool.

In this paper, we shall prove that the uniform convergence of solutions of difference-differential equations as the retardation approaches zero is a natural consequence of the continuity of solutions with respect to the retardation. The equation to be discussed here is rather general than that in their paper cited above. Among the variables appearing in the sequel, t represents a scalar and x, f may be vectors real or complex. By $|x|$ we denote a norm of x.

1. Continuity properties of solutions. We shall consider a difference-differential equation

(1.1) $$x'(t)=f(t, x(t), x(t-h))$$

for $0\leqq t\leqq t_0$, where h is a positive constant. The initial conditions imposed on (1.1) are that

(1.2) $$x(t)=\phi(t)\ (-h\leqq t\leqq 0) \quad \text{and} \quad x(0)=x_0,$$

where $\phi(t)$ is a given function continuous for $-h\leqq t\leqq 0$, $\phi(0)=x_0$, and $|\phi(t)-x_0|\leqq a$. Then, it is well known that if the function $f(t, x, y)$ satisfies the following conditions:

(i) $f(t,x,y)$ is continuous and $|f(t,x,y)|\leqq M$ for

(1.3) $$0\leqq t\leqq t_0,\quad |x-x_0|\leqq a,\quad |y-x_0|\leqq a;$$

(ii) $f(t,x,y)$ satisfies the Lipschitz condition, that is,

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\leqq k_1|x_1-x_2|+k_2|y_1-y_2|$$

for any t, x_i, y_i $(i=1,2)$ in the domain (1.3), where k_1 and k_2 are constants, then the existence and uniqueness of continuous solutions of (1.1) under the initial conditions (1.2) are established for $0\leqq t\leqq t^*=\min(t_0, a/M)$.

Now, for any positive constants h_i $(i=1,2)$ not greater than h, we consider two equations

(1.4) $$x'(t)=f(t, x(t), x(t-h_i))\quad (i=1,2)$$

*) R. Bellman and K. L. Cooke: On the limit of solutions of differential-difference equations as the retardation approaches zero, Proc. Nat. Acad. Sci., U.S.A., **45**, 1026–1028 (1959).

for $0 \leqq t \leqq t_0$ under the conditions

(1.5) $\quad x(t)=\phi(t)\ (-h_i \leqq t \leqq 0,\ i=1,2)$ and $x(0)=x_0$.

Since the unique solutions of (1.4) may be dependent on the retardation h_i, we denote them by $x(t, h_i)$ $(i=1,2)$ respectively. For the sake of brevity, it is supposed that $h_1 \leqq h_2$.

It follows by virtue of the initial conditions that

$$x(t, h_i)=x_0+\int_0^t f(t, x(t, h_i), \phi(t-h_i))dt \tag{1.6}$$

for $0 \leqq t \leqq h_i$, and

$$\begin{aligned} x(t, h_i)=x_0&+\int_0^{h_i} f(t, x(t, h_i), \phi(t-h_i))dt \\ &+\int_{h_i}^t f(t, x(t, h_i), x(t-h_i, h_i))dt \end{aligned} \tag{1.7}$$

for $h_i \leqq t \leqq t^*$. Now, we have to consider three cases:

I. The case $0 \leqq t \leqq h_1$. It follows from (1.6) and Lipschitz condition that

$$\begin{aligned} &|x(t, h_1)-x(t, h_2)| \\ &\leqq k_2\int_0^{h_1} |\phi(t-h_1)-\phi(t-h_2)|dt+k_1\int_0^t |x(t, h_1)-x(t, h_2)|dt. \end{aligned}$$

Then, it follows that

$$|x(t, h_1)-x(t, h_2)| \leqq k_2\int_0^{h_1} |\phi(t-h_1)-\phi(t-h_2)|dt\cdot\exp(k_1t^*). \tag{1.8}$$

II. The case $h_1 \leqq t \leqq h_2$. From (1.6), (1.7), and the hypotheses (i), (ii), it follows that

$$\begin{aligned} &|x(t, h_1)-x(t, h_2)| \\ &\leqq k_2\int_0^{h_1} |\phi(t-h_1)-\phi(t-h_2)|dt+2M(h_2-h_1)+k_1\int_0^t |x(t, h_1)-x(t, h_2)|dt. \end{aligned}$$

Then, it follows that

$$\begin{aligned} &|x(t, h_1)-x(t, h_2)| \\ &\leqq \left(k_2\int_0^{h_1} |\phi(t-h_1)-\phi(t-h_2)|dt+2M(h_2-h_1)\right)\exp(k_1t^*). \end{aligned} \tag{1.9}$$

III. The case $h_2 \leqq t \leqq t^*$. From (1.7) and the hypotheses (i), (ii), it follows that

$$\begin{aligned} &|x(t, h_1)-x(t, h_2)| \\ &\leqq (k_1+k_2)\int_0^t |x(t, h_1)-x(t, h_2)|dt+2M(h_2-h_1) \\ &\quad +k_2\int_0^{h_1} |\phi(t-h_1)-\phi(t-h_2)|dt \\ &\quad +k_2\int_{h_2}^t |x(t-h_1, h_1)-x(t-h_2, h_1)|dt. \end{aligned} \tag{1.10}$$

Since $|x'(t, h_1)|=|f(t, x(t, h_1), x(t-h_1, h_1))| \leqq M$ $(0 \leqq t \leqq t^*)$, we obtain by integrating that

$$|x(t-h_1, h_1)-x(t-h_2, h_1)| \leqq M(h_2-h_1) \quad (h_2 \leqq t \leqq t^*).$$

Hence, it follows from (1.10) that

$$|x(t, h_1)-x(t, h_2)| \leqq c_2+(k_1+k_2)\int_0^t |x(t, h_1)-x(t, h_2)|\,dt,$$

which implies that

$$|x(t, h_1)-x(t, h_2)| \leqq c_2 \exp((k_1+k_2)t^*), \tag{1.11}$$

where

$$c_2=M(h_2-h_1)(2+k_2(t^*-h_2))+k_2\int_0^{h_1} |\phi(t-h_1)-\phi(t-h_2)|\,dt.$$

Since $\phi(t)$ is continuous in $-h \leqq t \leqq 0$, for any given $\varepsilon>0$ there exists a $\delta>0$ such that $|\phi(t-h_1)-\phi(t-h_2)|<\varepsilon$ provided that $h_2-h_1<\delta$. Then, it follows from (1.8), (1.9), (1.11) that

$$|x(t, h_1)-x(t, h_2)| < K_1(h_2-h_1)+K_2\varepsilon \tag{1.12}$$

for $0 \leqq t \leqq t^*$, where K_1, K_2 are constants. The inequality (1.12) shows us that the solution $x(t, \lambda)$ is continuous for $0 \leqq \lambda \leqq h$. Thus, we have the following

THEOREM 1. *Suppose that in the equation* (1.1) *with* (1.2) $f(t, x, y)$ *satisfies the conditions* (i) *and* (ii). *Then, the unique solution* $x(t, h)$ *is a continuous function of* h.

2. The limit of solutions as $h \to 0$. Corresponding to the equation (1.1), we shall consider an ordinary differential equation

$$x'(t)=f(t, x(t), x(t)) \tag{2.1}$$

for $0 \leqq t \leqq t_0$ under the initial condition $x(0)=x_0$. Here, the function $f(t, x, y)$ is the same as in the preceding section.

The purpose of this section is to discuss whether or not the solution of (1.1) with (1.2) converges to the solution of (2.1). As was established above, the solution $x(t, h)$ is continuous in h. Furthermore, the inequality (1.12) implies that any family of solutions $\{x(t, h_n)\}_{n=1}^{\infty}$ where $h_n \to +0$ as $n \to \infty$, always has a Cauchy sequence and besides of that it is a bounded family. Then, by a well-known theorem of normal families, $x(t, h_n)$ converges uniformly to a function $x_0(t)$ continuous for $0 \leqq t \leqq t^*$ as $n \to \infty$. We may expect that $x_0(t)$ considered as the limit of the solution of (1.1) as $h \to +0$ is the solution of (2.1) for $0 \leqq t \leqq t^*$. This fact will be really established in the sequel by making use of the same method as before.

THEOREM 2. *Suppose that* $f(t, x, y)$ *satisfies the same conditions as in Theorem* 1. *Then, the solution* $x(t, h)$ *of* (1.1) *with* (1.2) *converges uniformly to the solution* $x(t)$ *of* (2.1) *for* $0 \leqq t \leqq t^*$, *as* h *tends to zero.*

PROOF. By means of the initial conditions, it follows from (1.1) and (2.1) that

$$x(t,h)=x_0+\int_0^t f(t,\ x(t,h),\ x(t-h,h))dt \tag{2.2}$$

and

$$x(t)=x_0+\int_0^t f(t,\ x(t),\ x(t))dt \tag{2.3}$$

for $0\leqq t\leqq t^*$. Since our situation is under the condition $h\to+0$, it is sufficient to suppose that $0<h<t^*$. It is necessary to consider two cases:

I. The case $0\leqq t\leqq h$. It follows from (2.2) and (2.3) that

$$|x(t,h)-x(t)|\leqq 2Mh. \tag{2.4}$$

II. The case $h\leqq t\leqq t^*$. It follows that

$$|x(t,h)-x(t)|\leqq 2Mh+(k_1+k_2)\int_0^t |x(t,h)-x(t)|dt+k_2Mh(t^*-h).$$

Hence, it follows that

$$|x(t,h)-x(t)|\leqq hM(2+k_2(t^*-h))\exp((k_1+k_2)t^*).$$

Together with (2.4), we have

$$|x(t,h)-x(t)|<k_3h,$$

where K_3 is a constant, which implies the uniform convergence of $x(t,h)$ to $x(t)$ for $0\leqq t\leqq t^*$ as $h\to+0$.

3. As to equations having many retardations, one of which tends to zero and the others are constants invariant, we can make use of the same method as stated above. For instance, we consider a difference-differential equation

$$x'(t)=f(t,\ x(t),\ x(t-h),\ x(t-1)) \quad (0<h<1) \tag{3.1}$$

for $0\leqq t\leqq t_0$ under the initial conditions

$$x(t)=\phi(t)\ (-1\leqq t\leqq 0) \quad \text{and} \quad x(0)=x_0, \tag{3.2}$$

where $\phi(t)$ is a given function continuous in $-1\leqq t\leqq 0$ and $\phi(0)=x_0$. The corresponding equation to (3.1) is

$$x'(t)=f(t,\ x(t),\ x(t),\ x(t-1)) \tag{3.3}$$

for $0\leqq t\leqq t_0$ with the initial conditions (3.2). Then, we have the following

THEOREM 3. *Suppose that in the equation* (3.1) $f(t,x,y,z)$ *satisfies the following conditions:*

(i) $f(t,x,y,z)$ *is continuous and* $|f(t,x,y,z)|\leqq M$ *for*

$$0\leqq t\leqq t_0,\quad |x-x_0|\leqq a,\quad |y-x_0|\leqq a,\quad |z-x_0|\leqq a; \tag{3.4}$$

(ii) $f(t,x,y,z)$ *satisfies the Lipschitz condition in the domain* (3.4).

Then, the unique solution of (3.1) *with* (3.2) *is a continuous function of h and converges uniformly to the unique solution of* (3.3) *with* (3.2) *for* $0\leqq t\leqq t^*=\min(t_0, a/M)$ *as h approaches zero.*