41. On Decomposition Theorems of the Vallée-Poussin Type in the Geometry of Parametric Curves

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1. **Introduction.** This is a continuation of our recent papers [1] to [3]. At the end of [2] we stated without proof a decomposition theorem for the measure-length induced by a locally rectifiable plane curve. It is the objective of the present note to prove this theorem in a slightly generalized form (see §4) and to obtain two further decomposition theorems concerning measure-length and spheric measure-length respectively. The last theorem will be applied elsewhere to derive a noteworthy property of the curvature of continuous parametric curves.

2. **Points of interjacence for a curve.** It is convenient to begin with two simple definitions. Let $\boldsymbol{R}^m$ be a Euclidean space of any dimension $m\geqq 2$ throughout the paper. Consider in $\boldsymbol{R}^m$ a parametric curve $\psi(t)$, defined and locally rectifiable on the real line $\boldsymbol{R}$. We shall term ψ *interjacent* at a point c of $\boldsymbol{R}$, if

$$|\psi(c-)-\psi(c+)|=|\psi(c-)-\psi(c)|+|\psi(c)-\psi(c+)|.$$

Further, a locally rectifiable, unit-spheric curve $\gamma(t)$ in $\boldsymbol{R}^m$ will be called *spherically interjacent* at c, if we have the angle-relation

$$\gamma(c-)\diamond\gamma(c+)=\gamma(c-)\diamond\gamma(c)+\gamma(c)\diamond\gamma(c+).$$

We may also call c *point of interjacence* of ψ and *point of spheric interjacence* of γ, in the respective cases.

The geometric meanings of the above two notions are easily seen. For instance, when $\psi(c-)\neq\psi(c+)$, the former notion means that the point $\psi(c)$ lies on the closed segment connecting the two points $\psi(c-)$ and $\psi(c+)$. (When the latter points coincide, interjacence of ψ at c is simply equivalent to its continuity at the same point.) We leave to the reader the consideration of the spheric case.

Evidently ψ [or γ] is interjacent [or spherically interjacent] wherever it is unilaterally (i.e. right-hand or left-hand) continuous.

3. **A lemma.** We shall now derive a result which includes the lemma left unproved in the final section of [3].

LEMMA. *Given ψ and γ as above, let S_* and Λ_* denote the measure-length induced by ψ and the spheric measure-length induced by γ, respectively. Then we have, for every point $t\in\boldsymbol{R}$,*

$$S_*(\{t\})=|\psi(t-)-\psi(t)|+|\psi(t)-\psi(t+)|,$$
$$\Lambda_*(\{t\})=\gamma(t-)\diamond\gamma(t)+\gamma(t)\diamond\gamma(t+).$$

PROOF. It suffices to ascertain the first relation, the second admitting a similar treatment. Let $s(t)$ be a length-function for the curve ψ, so that for every closed interval $[a, b]$ the length of ψ over $[a, b]$ is given by the increment $s(b)-s(a)$. Given an arbitrary positive number ε there is in the open interval $(t, t+\varepsilon)$ a point p such that $s(p)<s(t+)+\varepsilon$. Choose now a subdivision of the interval $[t, p]$ into closed intervals $J_0, J_1, \cdots, J_n (n\geqq 1)$ in such a way that $t\in J_0$ and that, moreover

$$s(p)-s(t)-\varepsilon<|\psi(J_0)|+\cdots+|\psi(J_n)|.$$

Since clearly $|\psi(J_1)|+\cdots+|\psi(J_n)|\leqq s(p)-s(t+)$, it follows that $|\psi(J_0)|$ exceeds $s(t+)-s(t)-\varepsilon$. Further $|\psi(J_0)|\leqq s(p)-s(t)<s(t+)-s(t)+\varepsilon$. By symmetry there exists in the half-open interval $(t-\varepsilon, t]$ a closed interval K_0 with right-hand extremity t, such that $s(t)-s(t-)-\varepsilon <|\psi(K_0)|<s(t)-s(t-)+\varepsilon$. In view of the evident relation $S_*(\{t\}) =s(t+)-s(t-)$ it follows that

$$S_*(\{t\})-2\varepsilon<|\psi(J_0)|+|\psi(K_0)|<S_*(\{t\})+2\varepsilon,$$

from which we deduce at once the desired equality by making $\varepsilon\to 0$. In fact $|\psi(J_0)|$ and $|\psi(K_0)|$ then tend respectively to $|\psi(t)-\psi(t+)|$ and $|\psi(t-)-\psi(t)|$, since $t\in J_0\subset[t, t+\varepsilon)$ and $t\in K_0\subset(t-\varepsilon, t]$.

4. Decomposition theorems for measure-length, of which the following first one extends slightly the theorem enunciated in [2]§5.

THEOREM. *Given an additive set-function μ, finite and nonnegative, and a locally rectifiable plane curve $\psi(t)=\langle x(t), y(t)\rangle$, with $s(J)$ for its arc length over closed intervals J, let C_0 be the set of the points of interjacence for ψ, and E_0 the Borel set of the points t at each of which one at least of the interior μ-derivatives $(\mu)x'(t)$ and $(\mu)y'(t)$ exists and is infinite. Then, for every bounded Borel set X in C_0,* we have

$$s^*(X)=s^*(E_0X)+\int_X\sqrt{[(\mu)x'(t)]^2+[(\mu)y'(t)]^2}\,d\mu(t). \tag{1}$$

PROOF. We shall write Θ for the outer measure s^* for brevity. It suffices to deal only with the two cases $X\subset C$ and $X\subset C_0-C$, where C means the set of the points of continuity of $\psi(t)$. Denoting further by H the set of the points t at each of which one or both of $(\mu)x'(t)$ and $(\mu)y'(t)$ exist and are infinite, and by A the set of the points t at which $(\mu)s'(t)=+\infty$, we have the evident relation $H\subset E_0\subset A$. Moreover $(\mu)\Theta'(t)=+\infty$ for every $t\in A$, since $\Theta(J)\geqq s(J)$ for all closed intervals J. On the other hand, if B stands for the set of the points t at which $(\mu)\Theta'(t)=+\infty$, then $(\mu)\Theta'(u)$ cannot exist at any point u of $A-B$. For otherwise we should get at once the contradiction $(\mu)\Theta'(u)=(\mu)\Theta'(u)=+\infty$. But $(\mu)\Theta'(t)$ exists almost everywhere (Θ) by part (i) of the decomposition theorem of [1]§8. We thus find $\Theta(A-B)=0$. This, combined with the relation $\Theta(BC-H)=0$ obtained in the course of the proof for the Supplement of [2]§4, yields us the equality $\Theta(AC-H)=0$. In view of $H\subset E_0\subset A$ it follows that $\Theta(E_0X)=\Theta(HX)$ provided that $X\subset C$. Consequently the case $X\subset C$ is reduced to the Supplement, *loc. cit.*

Let us consider the remaining case $X\subset C_0-C$. Since C_0-C is plainly countable, we may further restrict X to consist of a single point p. So that $x^*(X)$ and $y^*(X)$ cannot both vanish; indeed $x^*(X)=x(p+)-x(p-)$ and similarly for $y^*(X)$, as is well known. This being so, suppose firstly $\mu(X)=0$. Then either $(\mu)x'(p)=x^*(X)/\mu(X)=\pm\infty$, or $(\mu)y'(p)=\pm\infty$. Thus $p\in E_0$ and hence (1) is evident. On the other hand, if $\mu(X)\neq 0$, then again $(\mu)x'(p)=x^*(X)/\mu(X)$ and similarly for y, so that p does not belong to E_0. Therefore $\Theta(E_0X)=0$ and so (1) reduces to $\Theta(X)^2=x^*(X)^2+y^*(X)^2$, i.e. to $\Theta(X)=|\psi(p+)-\psi(p-)|$. But the last equation holds in virtue of the lemma of §3, since ψ is interjacent at p and since Θ coincides with the measure-length induced by ψ as shown in [3]§4. This completes the proof.

Theorem. *Let us write further* $F(J)=\sqrt{x(J)^2+y(J)^2}$ *for every closed interval J, in the above theorem. Then the points t of C_0 at which $(\mu)F'(t)$ and the μ-derivative of s^* exist and coincide, form a Borel set M such that*

$$\tilde{\mu}(C_0-M)=s^*(C_0-M)=0. \tag{2}$$

Furthermore, if E_1 denotes the Borel set of the points t at which $(\mu)F'(t)=+\infty$, we have for any bounded Borel set X in C_0

$$s^*(X)=s^*(E_1X)+\int_X(\mu)F'(t)d\mu(t). \tag{3}$$

Proof. We shall retain the notations of the foregoing proof. The functions $x(t)$ and $y(t)$, being of locally bounded variation, are both interior-derivable (μ) almost everywhere $(\tilde{\mu})$ (see [2]§5); while it is obvious that $(\mu)F'(t)=\sqrt{[(\mu)x'(t)]^2+[(\mu)y'(t)]^2}$ for every point t at which both x and y are interior-derivable (μ). Now $F(J)\leqq s(J)\leqq\Theta(J)$ for all closed intervals J, and Θ is μ-derivable almost everywhere $(\tilde{\mu})$ by Lebesgue's theorem of [1]§4. Consequently $(\mu)F'(t)\leqq(\mu)\Theta'(t)$ holds almost everywhere $(\tilde{\mu})$. On the other hand $E_0\subset E_1\subset A$, which together with the relation $\Theta(A-B)=0$ obtained in the proof of the foregoing theorem implies that

$$\Theta(E_0X)\leqq\Theta(E_1X)\leqq\Theta(AX)\leqq\Theta(BX)+\Theta(A-B)=\Theta(BX).$$

Combining all the above results we infer in view of (1) that

$$\Theta(X)\leqq\Theta(E_1X)+\int_X(\mu)F'(t)d\mu(t)\leqq\Theta(BX)+\int_X(\mu)\Theta'(t)d\mu(t). \tag{4}$$

But the last sum is equal to $\Theta(X)$ on account of the decomposition theorem of [1]§8. Thus the two signs of inequality in (4) may both be replaced by those of equality. This establishes the formula (3). Moreover the two integrals in (4) must coincide, and hence $\tilde{\mu}(C_0-M)$

$=0$ since X is arbitrary. It follows that $\Theta(Y)=\Theta(E_1Y)=\Theta(BY)$, where and subsequently Y is any bounded Borel set in C_0-M. Replacing Y by E_1Y here, we derive $\Theta(E_1Y)=\Theta(BE_1Y)=0$, since the intersection BE_1Y is clearly void. This implies that $\Theta(Y)=0$ for all Y and that therefore $\Theta(C_0-M)=0$. Thus (2) holds and the proof is complete.

5. Coincidence of ordinary and spheric measure-lengths for certain sets. Before proceeding to our third theorem we find it appropriate to interpose the following

LEMMA. *Given in $\boldsymbol{R}^m$ a locally rectifiable unit-spheric curve $\gamma(t)$, let S_* and Λ_* denote respectively the ordinary and spheric measure-lengths induced by γ. Then $S_*(X)=\Lambda_*(X)$ for every bounded Borel set X consisting exclusively of points of continuity for $\gamma(t)$.*

PROOF. It is enough to show that $S_*(X)\geqq\Lambda_*(X)$, the opposite inequality being obvious. Since $S_*(X)$ is the supremum of the values of S_* for closed subsets of X and similarly for $\Lambda_*(X)$, we may assume without loss of generality that X is a nonvoid closed set. Let D be any bounded open set containing X, so that $S_*(X)$ and $\Lambda_*(X)$ are respectively the infimum of $S_*(D)$ and that of $\Lambda_*(D)$. Let us keep D fixed for the moment and let ε be an arbitrary positive number. We shall denote by J closed intervals, and by $s(J)$ and $\lambda(J)$ the ordinary and spheric lengths of the curve γ over J respectively. By hypothesis we can associate with each point t of X a neighbourhood $N(t)$, i.e. an open interval with centre t, such that $\lambda(J)\leqq(1+\varepsilon)s(J)$ whenever $J\subset N(t)$. In fact we need only choose $N(t)$ so short that $\gamma(a)\diamond\gamma(b)\leqq(1+\varepsilon)\,|\,\gamma(J)\,|$ for every $J=[a,b]\subset N(t)$. We write $M(t)$ for the neighbourhood of t with length half as large as that of $N(t)$. In virtue of the Heine-Borel covering theorem there exists a finite nonvoid subset $\{t_1,\cdots,t_k\}$ of X such that the intervals $M(t_1),\cdots,M(t_k)$ together cover the set X. Denoting by 2δ the smallest of the lengths of these k intervals we easily see that if a closed interval J with length $<\delta$ intersects X, then necessarily $\lambda(J)\leqq(1+\varepsilon)s(J)$.

Now S_* and Λ_* coincide respectively with the outer measures s^* and λ^* induced by the additive interval-functions $s(J)$ and $\lambda(J)$, as remarked in [3](§4 and §5). We can therefore find in D a finite non-overlapping sequence $J_1,\cdots,J_n$ of closed intervals with lengths less than δ, in such a way that $\lambda(J_1)+\cdots+\lambda(J_n)>\Lambda_*(D)-\varepsilon$. We may suppose that the first p of these n intervals intersect X and the remaining ones do not, where p is some positive integer not exceeding n. Then $\lambda(J_i)\leqq(1+\varepsilon)s(J_i)$ for every $i=1,2,\cdots,p$ by what has already been proved, and consequently

$$\lambda(J_1)+\cdots+\lambda(J_p)\leqq(1+\varepsilon)S_*(D).$$

On the other hand it is clear that $\lambda(J_{p+1})+\cdots+\lambda(J_n)$ does not exceed $\Lambda_*(D-X)$ if $p<n$. Hence

$\Lambda_*(X) \leqq \Lambda_*(D) < \lambda(J_1) + \cdots + \lambda(J_n) + \varepsilon \leqq (1+\varepsilon) S_*(D) + \Lambda_*(D-X) + \varepsilon$. Since ε is arbitrary, this readily gives $2\Lambda_*(X) \leqq S_*(D) + \Lambda_*(D)$. So far the set D has been kept fixed. We now make it vary arbitrarily and obtain at once $2\Lambda_*(X) \leqq S_*(X) + \Lambda_*(X)$, i.e. $\Lambda_*(X) \leqq S_*(X)$, which completes the proof.

6. Decomposition theorem for spheric measure-length. The above lemma together with our second theorem enables us to deduce finally the following third

THEOREM. *Given an additive set-function μ, finite and nonnegative, and given in $\boldsymbol{R}^m$ a locally rectifiable unit-spheric curve $\gamma(t)$, let Q be the set of the points of spheric interjacence for γ, and write $G(J) = \gamma(a) \diamond \gamma(b)$ for every closed interval $J = [a, b]$. Then the points t of Q at which $(\mu)G'(t)$ and $(\mu)\Lambda_*'(t)$ exist and coincide, form a Borel set N such that*

$$\tilde{\mu}(Q-N) = \Lambda_*(Q-N) = 0, \tag{5}$$

where Λ_* means as before the spheric measure-length determined by γ.

Furthermore, if H denotes the Borel set of the points t at which $(\mu)G'(t) = +\infty$, we have for any bounded Borel set X in Q

$$\Lambda_*(X) = \Lambda_*(HX) + \int_X (\mu)G'(t) d\mu(t). \tag{6}$$

PROOF. Write $F(J) = |\gamma(J)|$ for closed intervals J and let C be the set of the points of continuity of $\gamma(t)$. It is readily seen that if one of $(\mu)F'(t)$ and $(\mu)G'(t)$ exists at a point t of C, then the other also exists and the two values are equal. Thus $(\mu)G'(t)$ exists almost everywhere $(\tilde{\mu})$ in C, since the same is true of $(\mu)F'(t)$ on account of (2). Suppose now that $X \subset C$. Then the preceding lemma gives $\Lambda_*(X) = S_*(X)$ as well as $\Lambda_*(HX) = S_*(HX)$, and the formula (6) is a direct consequence of (3). (In §4 we only considered plane curves. But needless to say, this restriction is not essential for the validity of the two theorems of that section.)

This being so, let p be any point of $Q-C$, so that $\gamma(p+) \neq \gamma(p-)$. It is easy to see that $(\mu)G'(p)$ exists and is given by

$$(\mu)G'(p) = \{\gamma(p-) \diamond \gamma(p+)\} / \mu(\{p\}). \tag{7}$$

If in particular $\mu(\{p\})$ vanishes here, then $(\mu)G'(p) = +\infty$; so that $p \in H$ and hence (6) is manifest when the set X is specialized to $\{p\}$. If on the other hand $\mu(\{p\}) \neq 0$, then (7) shows $(\mu)G'(p)$ finite; so that p does not belong to H and accordingly (6) reduces for $X = \{p\}$ to $\Lambda_*(X) = \gamma(p-) \diamond \gamma(p+)$. But the last equation is true by virtue of the lemma of §3, since the curve γ is spherically interjacent at p by hypothesis. The set $Q-C$ being evidently countable, we have thus verified that (6) holds for $X \subset Q-C$.

Now the case of general X is readily reduced to the two cases $X \subset C$ and $X \subset Q-C$ treated already, by expressing X as the join of

XC and $X-C$. This proves (6) completely. As we may observe, we have also proved that $(\mu)G'(t)$ exists almost everywhere $(\tilde{\mu})$ in Q. It follows furthermore from (6) that $\tilde{\mu}(HQ)$ vanishes.

It remains to derive (5). For this purpose we write K for the Borel set of the points t at which we have $(\mu)\Lambda'_*(t)=+\infty$, and find by the decomposition theorem of [1]§8 that, for every bounded Borel set X in Q,

$$(8) \qquad \Lambda_*(X)=\Lambda_*(KX)+\int_X (\mu)\Lambda'_*(t)d\mu(t).$$

Replacing X by HX here, we deduce in view of $\tilde{\mu}(HQ)=0$ that $\Lambda_*(HX)=\Lambda_*(HKX)$. Similarly we obtain $\Lambda_*(KX)=\Lambda_*(HKX)$ from (6) with the help of the relation $\tilde{\mu}(K)=0$, which holds on account of Lebesgue's theorem of [1]§4. It follows readily that the integral in (8) coincides with that in (6). From this we infer easily that $\tilde{\mu}(Q-N)=0$ for the set N of the assertion. Consider now the special case in which $X\subset Q-N$. Then (8) reduces to $\Lambda_*(X)=\Lambda_*(KX)$. Combining the last equation with $\Lambda_*(KX)=\Lambda_*(HKX)$ proved already, in which the intersection HKX must be void by definition of the set N, we get at once $\Lambda_*(X)=0$. It follows finally that $\Lambda_*(Q-N)=0$, which completes the proof of our theorem.

References

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