58. On Open Mappings. II

By Sitiro HANAI

Osaka University of Liberal Arts and Education

(Comm. by K. KUNUGI, M.J.A., May 13, 1961)

Let X and Y be topological spaces and let f be a continuous mapping of X onto Y. f is said to be open if the image of every open subset of X is open in Y. A. H. Stone [9] has obtained conditions under which the image of an open continuous mapping of a metric space becomes metrizable. In this note, we shall obtain some results concerning the images of the open continuous mappings of metric spaces.

1. By the *open image*, we mean the image of an open continuous mapping. We begin with proving the following theorem.

Theorem 1. *If X is a T_1-space which satisfies the first countability axiom, then X is an open image of a metric space.*

Proof. Let $\{U_\alpha \mid \alpha \in \Omega\}$ be the open basis of X. For each point x of X, let $\{U_{\alpha_i} \mid i=1,2,\cdots;\ \alpha_i \in \Omega\}$ be an open neighborhood basis of x, then $\alpha=(\alpha_1,\alpha_2,\cdots)\in N(\Omega)$, where $N(\Omega)$ is the generalized Baire's zero-dimensional space*) introduced by K. Morita [4]. Now let A denote the set of all such α. If we define a mapping f of A into X by $f(\alpha)=x$, then it is evident that $f(A)=X$. We shall next prove that f is an open continuous mapping. Let V be any open neighborhood of x such that $f(\alpha)=x$, then, since $\{U_{\alpha_i} \mid i=1,2,\cdots\}$ is an open neighborhood basis of x, there exists a U_{α_k} such that $U_{\alpha_k}\subset V$. Then if $\rho(\alpha,\beta)<\frac{1}{k}$ where $\beta=(\beta_1,\beta_2,\cdots)\in A$, then $\alpha_i=\beta_i$ for $i\leqq k$ by the definition of the metric of $N(\Omega)$. Hence $f(\beta)\in\bigcap_{i=1}^{k}U_{\alpha_i}\subset U_{\alpha_k}\subset V$. Therefore f is continuous.

Now let $V\left(\alpha;\frac{1}{k}\right)=\left\{\beta \mid \rho(\alpha,\beta)<\frac{1}{k}\right\}$, then $f\left(V\left(\alpha;\frac{1}{k}\right)\right)=\bigcap_{i=1}^{k}U_{\alpha_i}$. In fact, since $f\left(V\left(\alpha;\frac{1}{k}\right)\right)\subset\bigcap_{i=1}^{k}U_{\alpha_i}$, it is sufficient to show that $f\left(V\left(\alpha;\frac{1}{k}\right)\right)\supset\bigcap_{i=1}^{k}U_{\alpha_i}$. For this purpose, let $y\in\bigcap_{i=1}^{k}U_{\alpha_i}$ and let $\{U_{\beta_j} \mid j=k+1, k+2,\cdots\}$ be an open neighborhood basis which is obtained by number-

*) We define the metric ρ of $N(\Omega)=\{(\alpha_1,\alpha_2,\cdots) \mid \alpha_i\in\Omega, i=1,2,\cdots\}$ as follows: if $\alpha=(\alpha_1,\alpha_2,\cdots)$, $\beta=(\beta_1,\beta_2,\cdots)$, $\alpha_i=\beta_i$ for $i<n$, $\alpha_n\neq\beta_n$, then $\rho(\alpha,\beta)=\frac{1}{n}$. As is well known, $N(\Omega)$ is a 0-dimensional metric space and we call $N(\Omega)$ a generalized Baire's zero-dimensional space according to K. Morita.

ing from $k+1$. Then $\{U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_k}, U_{\beta_{k+1}}, \cdots\}$ is an open neighborhood basis of y. Hence $f(\alpha')=y$ where $\alpha'=(\alpha_1,\cdots,\alpha_k,\beta_{k+1},\cdots)$. Therefore we get $f\left(V\left(\alpha;\frac{1}{k}\right)\right)\supset\bigcap_{i=1}^{k}U_{\alpha_i}$. On the other hand, since $\left\{V\left(\alpha;\frac{1}{k}\right)\,\middle|\,\alpha\in A, k=1,2,\cdots\right\}$ is the basis for open sets of A, f is an open mapping. This completes the proof.

The following theorem is an immediate consequence of Theorem 1.

Theorem 2. *A T_1-space X is an open image of a metric space if and only if X satisfies the first countability axiom.*

Theorem 3. *A regular space X is an open image of a locally separable metric space if and only if X is locally separable and locally metrizable.*

Proof. Let $f(T)=X$ be an open continuous mapping where T is a locally separable metric space. Let U be an open separable neighborhood of any point t of T, then, since f is open, $f(U)$ is an open set in X and a regular space with a countable basis for open sets as a subspace of X. Hence $f(U)$ is separable and metrizable as a subspace of X. Thus X is locally separable and locally metrizable.

Conversely, if X is locally separable and locally metrizable, then, for each point x of X, there exists an open neighborhood $U(x)$ which is separable and metrizable. Let V_x be a topological space such that V_x is homeomorphic to $U(x)$ and $V_x\cap V_{x'}=\phi$ for any distinct points x and x' of X, then $T=\bigcup_{x\in X}V_x$ is a locally separable metric space where the topology of T is defined as follows: for each point t of T such that $t\in V_x$, the open neighborhood basis of t is the open neighborhood basis of t of the space V_x. Let φ_x be the above homeomorphism between $U(x)$ and V_x and let $f(t)=\varphi_x(t)$ if $t\in V_x$. Then it is easy to see that f is an open continuous mapping of T onto X. This completes the proof.

By Theorem 3, we easily obtain the following theorem.

Theorem 4. *Let X be a locally separable metric space and let Y be a regular space. If $f(X)=Y$ is an open continuous mapping, then Y is metrizable if and only if Y is paracompact.*

Remark 1. In Theorem 4, we assumed that X is locally separable. We can not drop this assumption since we can get an encounter example as follows. Let R be a set of real numbers. For each $x\in R$, we define an open neighborhood basis $\{U_u(x)\mid x<u, u\in R\}$ where $U_u(x)=\{y\mid x\leqq y<u\}$. Then R is a paracompact Hausdorff space satisfying the first countability axiom, but R is not metrizable. By Theorem 1, R is an open image of a metric space.

Remark 2. By E. Michael's theorem [2, Cor. 1.4], we can prove the following theorem: *let X be a complete metric space and let Y*

be a zero-dimensional paracompact T_1-space. If Y is an open image of X, then Y is metrizable.

In conclusion of this section, we shall prove the following theorem. Before the proof, we give a definition. Let $f(X)=Y$ be a continuous mapping. If $f^{-1}(y)$ is compact for each point y of Y, then Y is said to be a *compact image.*

Theorem 5. *A Hausdorff space X is an open compact image of a metric space if and only if there exists a countable family $\{\mathfrak{V}_i\}$ of point-finite open coverings of X such that $\{S(x, \mathfrak{V}_i)\} \mid i=1, 2\cdots\}$ is a neighborhood basis of x for each point x of X.*

Proof. Necessity. Let $f(T)=X$ be an open continuous mapping of a metric space T onto X such that $f^{-1}(x)$ is compact for each $x \in X$. Let $\mathfrak{G}_n=\{G^n\}$ be a locally finite open covering of T such that the diameter of G^n is less than $\frac{1}{n}$ for each $G^n \in \mathfrak{G}_n$. Then $\mathfrak{V}_n=f(\mathfrak{G}_n)$ $=\{f(G^n) \mid G^n \in \mathfrak{G}_n\}$ is a point-finite open covering of X because f is open and $f^{-1}(x)$ is compact for each $x \in X$. Let V be any open neighborhood of x, then $f^{-1}(x) \subset f^{-1}(V)$. Since $f^{-1}(x)$ is compact, $\rho[f^{-1}(x), T-f^{-1}(V)]>0$ where ρ denotes the metric of T. If we find a positive integer m such that $\rho[f^{-1}(x), T-f^{-1}(V)]>\frac{1}{m}$, then $S(f^{-1}(x), \mathfrak{G}_m) \subset f^{-1}(V)$. Hence $S(x, \mathfrak{V}_m) \subset V$.

Sufficiency. Let $\mathfrak{V}_i=\{V_\alpha^i \mid \alpha \in \Gamma_i\}$ $(i=1, 2,\cdots)$ and let $P=\prod_{i=1}^{\infty}\Gamma_i$, that is, P is the product of the sets Γ_i. Let T be the set of elements $(\alpha_1, \alpha_2,\cdots)$ of P such that $\bigcap_{i=1}^{\infty} V_{\alpha_i}^i$ is a single point of X. Now we define a mapping f as follows: $f(\alpha_1, \alpha_2,\cdots)=\bigcap_{i=1}^{\infty} V_\alpha^i$. Then it is easy to see that $f(T)=X$. If we consider T as the subspace of the generalized Baire's zero-dimensional space, then T is a metric space. We shall next prove that f is an open continuous mapping. To show this, let V be any open neighborhood of x, there exists a k such that $S(x, \mathfrak{V}_k) \subset V$. If we take an open neighborhood $V\left(\alpha ; \frac{1}{k}\right)=\{(\beta_1, \beta_2, \cdots)$ $\mid \beta_i=\alpha_i, i=1, 2,\cdots, k\}$ of $\alpha=(\alpha_1, \alpha_2,\cdots)$, we get $x \in f\left(V\left(\alpha ; \frac{1}{k}\right)\right) \subset S(x, \mathfrak{V}_k) \subset V$. Hence f is continuous. We have also $f\left(V\left(\alpha ; \frac{1}{k}\right)\right)=\bigcap_{i=1}^{\infty} V_{\alpha_i}^i$. In fact, since $f\left(V\left(\alpha ; \frac{1}{k}\right)\right) \subset \bigcap_{i=1}^{k} V_{\alpha_i}^i$, we need only show that $f\left(V\left(\alpha ; \frac{1}{k}\right)\right) \supset \bigcap_{i=1}^{k} V_{\alpha_i}^i$. Let x be any point of $\bigcap_{i=1}^{k} V_{\alpha_i}^i$, then there exists an l such that $S(x, \mathfrak{V}_l) \subset \bigcap_{i=1}^{k} V_{\alpha_i}^i$ and $l>k$. Then there exists a $\gamma=(\alpha_1,\cdots,$

$\alpha_k, \gamma_{k+1}, \gamma_{k+2}, \cdots)$ such that $x=(\bigcap_{i=1}^{k} V^i_{\alpha_i})\frown(\bigcap_{j=k+1}^{\infty} V^j_{\gamma_j})$. Hence $f(\gamma)=x$ and $\gamma \in V\left(\alpha ; \frac{1}{k}\right)$. Therefore $f\left(V\left(\alpha ; \frac{1}{k}\right)\right)\supset\bigcap_{i=1}^{k} V^i_{\alpha_i}$. On the other hand, since $\left\{V\left(\alpha ; \frac{1}{k}\right) \middle| k=1, 2, \cdots, \alpha \in T\right\}$ is a basis for open sets of T, f is open. Finally we prove that $f^{-1}(x)$ is compact for each point x of X. Let $\varDelta_i$ be the set of α_i such that $x \in V^i_{\alpha_i}$, then, by the assumption, $\varDelta_i$ is a finite set. Hence $f^{-1}(x)=\prod_{i=1}^{\infty} \varDelta_i$ is compact. This completes the proof.

2. In the first place, we shall give an another proof of the following well-known theorem due to A. H. Stone [9].

Theorem 6 (A. H. Stone). *If f is an open continuous mapping of a metric, locally separable space X onto a regular space Y, and if for each point y of Y the set $f^{-1}(y)$ is separable, then Y is metrizable and locally separable.*

Proof. Let U be an open separable subset of X, then $f(U)$ is separable and metrizable since Y is a regular space.

Now let $\mathfrak{H}=\{H\}$ be any open covering of Y, then $f^{-1}(\mathfrak{H})=\{f^{-1}(H) \mid H \in \mathfrak{H}\}$ is an open covering of X. Since X is a metric space and locally separable, X has the star-finite property [3, Cor. 2, p. 67]. Hence there exists a star-finite open refinement $\mathfrak{G}=\{G\}$ of $f^{-1}(\mathfrak{H})$ such that each G is separable. Then $f(\mathfrak{G})=\{f(G) \mid G \in \mathfrak{G}\}$ is a star-countable open refinement of $\mathfrak{H}$ since $f(G)\frown f(G')\neq\phi$ if and only if $G \frown f^{-1}(f(G'))\neq\phi$ and $f^{-1}(f(G))$ is separable. Thus we can see that Y is a regular space with the star-countable property. Therefore Y has the star-finite property [8]. Hence Y is paracompact. Thus we get the theorem by use of Theorem 4, completing the proof.

Theorem 7. *If a collectionwise normal space Y is an open compact image of a metric space, then Y is metrizable.*

Proof. Let X be a metric space and let $f(X)=Y$ be an open continuous mapping such that $f^{-1}(y)$ is compact for each point y of Y. Let $\mathfrak{G}_n$ denote the locally finite open covering of X defined in the proof of Theorem 5. Then $f(\mathfrak{G}_n)=\{f(G^n) \mid G^n \in \mathfrak{G}_n\}$ is a point-finite open covering of Y. Since Y is collectionwise normal, by E. Michael's theorem [1] $f(\mathfrak{G}_n)$ has a locally finite open refinement $\mathfrak{H}_n=\{H\}$. Let y be any point of Y and let V be any open neighborhood of y. Then, since $f^{-1}(y)$ is compact, $\rho[f^{-1}(y), X-f^{-1}(V)]>0$ where ρ is the metric of X. Hence there exists a positive integer m such that $\rho[f^{-1}(y), X-f^{-1}(V)]>\frac{1}{m}$. Then $S(f^{-1}(y), \mathfrak{G}_m)\subset f^{-1}(V)$. Hence $S(y, f(\mathfrak{G}_m))\subset V$. Therefore we get $S(y, \mathfrak{H}_m)\subset S(y, f(\mathfrak{G}_m))\subset V$. By Nagata-Smirnov's theorem [6, 7], Y is metrizable. This completes the proof.

Remark 3. i) If we drop the condition that $f^{-1}(y)$ is compact for each point y of Y, Theorem 7 does not hold. We can see this by the example given by A. H. Stone [9, Example (1)].

ii) In Theorem 7, we can not replace the collectionwise normality of Y by the regularity. This is also seen by A. H. Stone [9, Example (3)].

Theorem 8. *Let X be a paracompact topological space and let Y be a collectionwise normal space. If $f(X)=Y$ is an open continuous mapping such that the boundary $\mathfrak{B}f^{-1}(y)$ is compact for each point y of Y, then Y is paracompact.*

Proof. Let $Y_0=\{y \mid \text{Int } f^{-1}(y)\neq\phi\}$, then by the openness of f, $f(\text{Int } f^{-1}(y_0))=y_0$ is open for each point y_0 of Y. Therefore Y_0 is a discrete open subspace and $Y-Y_0$ is closed. Let $X_0=f^{-1}(Y_0)$, then $X-X_0$ is a closed inverse set and paracompact as the subspace of X. Let f_0 denote the mapping f of $X-X_0$ onto $Y-Y_0$, then f_0 is open and continuous. Since $\text{Int } f^{-1}(y)=\phi$ for each point y of $Y-Y_0$, we get $\mathfrak{B}f^{-1}(y)=f^{-1}(y)$. Then we can see that $\mathfrak{B}f_0^{-1}(y)=f_0^{-1}(y)=f^{-1}(y)$, that is, the boundary of $f_0^{-1}(y)$ in the subspace $X-X_0$ coincides with the compact set $f^{-1}(y)$. We shall next prove the paracompactness of the subspace $Y-Y_0$. Let $\mathfrak{H}=\{H\}$ be any open covering of $Y-Y_0$. Then we have the open covering $f^{-1}(\mathfrak{H})=\{f^{1}(H) \mid H\in\mathfrak{H}\}$ of $X-X_0$. Since $X-X_0$ is paracompact, there exists a locally finite open refinement $\mathfrak{G}=\{G\}$ of $f^{-1}(\mathfrak{H})$. Then $f(\mathfrak{G})=\{f(G) \mid G\in\mathfrak{G}\}$ is a point-finite open covering of $Y-Y_0$ because f is open and $f_0^{-1}(y)$ is compact for each point y of $Y-Y_0$. On the other hand, $Y-Y_0$ is collectionwise normal by the closedness of $Y-Y_0$. Therefore, by E. Michael's theorem [1], $f(\mathfrak{G})$ has a locally finite open refinement. Hence $Y-Y_0$ is paracompact. Therefore, by K. Morita's lemma [5, Lemma 1], Y is paracompact. This completes the proof.

Theorem 9. *Let X be a locally countably compact paracompact topological space and let Y be a Hausdorff space satisfying the first countability axiom. If $f(X)=Y$ is an open continuous mapping such that the inverse image of any countably compact set of Y under f is compact, then Y is paracompact.*

Proof. Let $\mathfrak{G}=\{G_\alpha\}$ be an open covering of Y, then $\mathfrak{G}'=\{f^{-1}(G_\alpha) \mid G_\alpha\in\mathfrak{G}\}$ is an open covering of X. Since X is paracompact, $\mathfrak{G}'$ has a locally finite open refinement $\mathfrak{R}=\{R_\beta\}$. Then $\mathfrak{R}'=\{f(R_\beta) \mid R_\beta\in\mathfrak{R}\}$ is an open refinement of $\mathfrak{G}$. It is easy to see that $\mathfrak{R}'$ is point-finite since $f^{-1}(y)$ is compact for each point y of Y by the assumption.

Now we shall prove that $\mathfrak{R}'$ is locally finite. For this purpose, we suppose on the contrary that $\mathfrak{R}'$ is not locally finite. On the other hand, since X is locally countably compact, Y is also locally countably compact. Then there exist a point y_0 and a countably compact neighborhood $U(y_0)$ of y_0 such that $U(y_0)$ intersects infinitely many sets of $\mathfrak{R}'$. Let y_1 be a point of $U(y_0)$, then y_1 is contained in only a finite number of sets $f(R_\beta)$. We denote by $\mathfrak{R}'(y_1)$ the family of all such sets $f(R_\beta)$. We can find a point y_2 of $U(y_0)$ such that $\mathfrak{R}'(y_1)\frown\mathfrak{R}'(y_2)=\phi$ since $U(y_0)$ intersects infinitely many sets of $\mathfrak{R}'$ and $\mathfrak{R}'$ is point-finite. In the same way, we can find a point y_3 of $U(y_0)$ such that $\mathfrak{R}'(y_3)\frown(\mathfrak{R}'(y_1)\smile\mathfrak{R}'(y_2))=\phi$. By induction, we get a sequence $\{y_n\}$ of points of $U(y_0)$ such that $\mathfrak{R}'(y_n)\frown\mathfrak{R}'(y_m)=\phi$ for $n\neq m$. Since $U(y_0)$ is countably compact and Y satisfies the first countability axiom, we may assume that $\{y_n\}$ converges to a point y of $U(y_0)$. Since Y is a Hausdorff space, the set $\{y_n\}\smile\{y\}$ is countably compact. Hence the set $(\bigcup_{n=1}^{\infty} f^{-1}(y_n))\smile f^{-1}(y)$ intersects only a finite number of sets of $\mathfrak{R}$. Thus we get a contradiction, completing the proof.

References

[1] E. Michael: Point-finite and locally finite coverings, Canad. Jour. Math., **7**, 275–279 (1955).

[2] ——: Continuous collections II, Ann. of Math., **64**, 562–580 (1956).

[3] K. Morita: Star-finite coverings and the star-finite property, Math. Japonicae, **1**, 60–68 (1948).

[4] ——: A condition for the metrizability of topological spaces and for n-dimensionality, Sci. Rep. Tokyo Kyoiku Daigaku, sec. A, **5**, no. 114, 33–36 (1955).

[5] ——: On closed mappings, Proc. Japan Acad., **32**, 539–543 (1956).

[6] J. Nagata: On a necessary and sufficient condition of metrizability, Jour. Inst. Polytech. Osaka City Univ., ser. A, **1**, 93–100 (1950).

[7] Yu. M. Smirnov: A necessary and sufficient condition for metrizability of topological spaces, Dokl. Akad. Nauk SSSR, **77**, 197–200 (1951).

[8] ——: On strongly paracompact spaces, Izv. Nauk SSSR, Ser. Mat., **20**, 253–274 (1956).

[9] A. H. Stone: Metrizability of decomposition spaces, Proc. Amer. Math. Soc., **7**, 690–700 (1956).