

57. On Two Properties of the Curvature of Continuous Parametric Curves

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1. Introduction. In the present continuation of his recent papers [1] to [5] the author proceeds to establish two noteworthy properties of the curvature of continuous parametric curves. They generalize certain well-known results in classical differential geometry of curves.

Let R^m be a Euclidean space of any dimension $m \geq 2$ throughout this note. Let us consider in this space a parametric curve $\varphi(t)$ of the class C^2 , defined and regular on the real line R . In other words, we suppose that the coordinate-functions $x_i(t)$ of φ are all twice continuously differentiable ($i=1, 2, \dots, m$) and that, furthermore, the derivative of φ , given by $\varphi'(t) = \langle x_1'(t), \dots, x_m'(t) \rangle$ for all t , never vanishes. Let $s(t)$ denote a length-function for the curve φ , so that $s(t)$ increases strictly and for every closed interval $[a, b]$ the arc-length of φ over $[a, b]$ is equal to the increment $s(b) - s(a)$. We write further $\gamma(t)$ for the spheric representation of φ , given by $\gamma(t) = |\varphi'(t)|^{-1} \varphi'(t)$ for each t . Then everybody knows that the curvature of φ at any point t of R is expressed by the absolute value of the s -derivative $(s)\gamma'(t)$ of the curve γ . Indeed this is often adopted as the definition of curvature.

Now the extension of this statement to curves more general than φ considered above is the concern of our first theorem (§3). It should be noted that in our paper [4] we defined curvature in a way different from the aforesaid standard definition and that therefore the propounded extension is not a definition but a theorem requiring a regular proof. As for our second theorem (§5), we must omit the explanation of its origin owing to space limitation.

2. Direction-curves. Consider in R^m a continuous light curve $\varphi(t) = \langle x_1(t), \dots, x_m(t) \rangle$, defined on R and locally straightenable (see [4] §2). Then φ is necessarily locally rectifiable by [1] §64. As in [4], the length and bend of φ over an interval I (of any type) will be denoted by $S(I)$ and $\Omega(I)$ respectively, and the induced measure-length and measure-bend by S_* and Ω_* respectively. Further, we shall continue using the symbol $\rho(t)$ of [4] to denote the curvature (in our sense) of φ at a point t of R . We remark in passing that, as easily seen, the definition of bend adopted in [4] is compatible

with that of [1]§28. Since φ is light, there exists at least one direction-curve for φ (in the sense of [1]§44) defined on the whole real line. We denote by $\gamma(t)$ any fixed one of them, and by Λ and Λ_* respectively its spheric length and spheric measure-length (*vide* [4]§5).

LEMMA. *The direction-curve γ is locally rectifiable and spherically interjacent (cf. [5]§2). Moreover $\Omega(I)=\Lambda(I)$ for every endless interval I , so that the two outer measures Ω_* and Λ_* are identical.*

PROOF. We shall write $\varphi^R(t)$ and $\varphi^L(t)$ for the right-hand and left-hand spheric representations of φ respectively as in [1]§77, their existence being ensured by [1]§80. Denoting by T the set of the points t satisfying $\varphi^R(t)\neq\varphi^L(t)$, we find immediately $\Omega(I)\geq\varphi^L(p)\diamond\varphi^R(p)$ for every point p of T and every endless interval I containing p . Consequently $\Omega_*({p})$ never vanishes for $p\in T$, and so T must be a countable set. On the other hand, the curves γ and φ^R are both locally rectifiable on account of [1]§45. The proposition of [1]§83 then shows that $\gamma(t+)=\varphi^R(t)$ and $\gamma(t-)=\varphi^L(t)$ for all t of \mathbf{R} and that φ^R is right-hand continuous.

In order to ascertain now the spheric interjacency of the curve γ , suppose that $\gamma(c)$ coincides with neither of $\gamma(c-)$ and $\gamma(c+)$, where c is a fixed point of \mathbf{R} . By definition of $\gamma(c)$ we can choose an infinite sequence J_1, J_2, \dots of closed intervals containing c and tending to c , such that $\varphi(J_n)\neq 0$ for all $n=1, 2, \dots$ and such that $\varphi(J_n)\diamond\gamma(c)\rightarrow 0$ as $n\rightarrow\infty$. Writing $J_n=[a_n, b_n]$ for each n , we find the existence of a natural number n_0 such that $\varphi(a_n)\neq\varphi(c)$ and $\varphi(b_n)\neq\varphi(c)$ for every n exceeding n_0 . In fact, if for instance $\varphi(a_n)=\varphi(c)$ were true for all n belonging to some infinite set N of natural numbers, we should have $\varphi(b_n)-\varphi(c)=\varphi(J_n)$ for $n\in N$. Then, as n tends to infinity in the set N , the equality

$$\{\varphi(b_n)-\varphi(c)\}\diamond\gamma(c)=\varphi(J_n)\diamond\gamma(c)$$

would at once yield the contradiction $\gamma(c+)\diamond\gamma(c)=\varphi^R(c)\diamond\gamma(c)=0$. This being so, write for short $P_n=[a_n, c]$ and $Q_n=[c, b_n]$, where and subsequently we suppose $n>n_0$. Making $n\rightarrow\infty$ in the triangular equality (see [1]§23)

$$\varphi(P_n)\diamond\varphi(Q_n)=\varphi(P_n)\diamond\varphi(J_n)+\varphi(Q_n)\diamond\varphi(J_n),$$

we get at once the following relation which shows γ spherically interjacent:

$$\gamma(c-)\diamond\gamma(c+)=\gamma(c-)\diamond\gamma(c)+\gamma(c+)\diamond\gamma(c).$$

Return now to the set T considered at the beginning. The last equation implies that $\gamma(t)=\gamma(t+)=\varphi^R(t)$ for every t not belonging to T . Arguing as in §§84-85 of [1] we then see without difficulty that, for every endless interval I , the spheric length of the curve φ^R over I equals $\Lambda(I)$. This combined with [1]§96 leads to $\Omega(I)=\Lambda(I)$, com-

pleting the proof.

3. **First theorem.** With the help of the above lemma we go on now to drive the following result which is the first of our two theorems:

THEOREM. *Given φ and γ as in the foregoing section, with S and Ω for the length and bend of φ respectively, write $F(J)=\gamma(a)\diamond\gamma(b)$ for closed intervals $J=[a, b]$ and denote by K the Borel set of the points t of \mathbf{R} for which the curvature $\rho(t)$ of φ equals $(S)F''(t)$, i.e. the interior S -derivative of F at t . Then we have*

$$S_*(\mathbf{R}-K)=\Omega_*(\mathbf{R}-K)=0.$$

PROOF. As the theorem of [4]§7 asserts, $\rho(t)$ is synonymous with $(S_*)\Omega'_*(t)$, where Ω_* coincides identically with Λ_* in conformity with the above lemma. Further $S_*(J)=S(J)$ for all closed intervals J in virtue of continuity of φ . Accordingly, in view of local rectifiability and spheric interjacency of γ (see the lemma), the set K of the assertion coincides with the set N considered in the theorem of [5]§6, provided that we take there for the additive set-function μ the restriction of S_* to bounded Borel sets. But $\tilde{\mu}$ then coincides with S_* and so our theorem is equivalent to the relation (5) of the theorem quoted just now.

4. **Second order approximation to an indefinite integral.** As in our previous papers, the letter μ will always stand for a non-negative additive set-function. Let c be a given point of \mathbf{R} and, for the moment, let us denote by J any closed interval for which c is an extremity. We shall say that a finite interval-function P , defined for all closed intervals, admits *second order approximation* around the point c with respect to μ , when the following condition is fulfilled: there exist a pair A, B of finite real coefficients and a finite real function $\vartheta(J)$ of the interval J in such a manner that $\vartheta(J)\rightarrow 0$ as J tends to c and that further, for every J ,

$$P(J)=A\mu(J)+B\xi(J)\mu^2(J)+\vartheta(J)\mu^2(J),$$

where $\xi(J)$ means 1 or -1 according as c is the left-hand or right-hand extremity of J . It should be observed that, as a matter of course, the coefficients A, B and the function $\vartheta(J)$ may depend not only on μ and P , but also on the point c . We can also express the remainder term $\vartheta(J)\mu^2(J)$ by the Landau notation $o[\mu^2(J)]$. Of course the above concept will mostly be used when μ is continuous at c , that is, when $\mu(\{c\})$ vanishes.

Consider now on \mathbf{R} a finite point-function $f(t)$ which is B-measurable and locally integrable (μ). The latter condition means that the function is integrable (μ) over every finite interval, or what amounts to the same thing, over every bounded Borel set. The indefinite μ -integral of $f(t)$ over closed intervals I will be denoted by $\Phi(I)$ for

simplicity.

LEMMA. Given μ , $f(t)$, and $\Phi(I)$ as above, suppose that μ is continuous, i.e. that $\mu(\{t\})=0$ for all points t . Then Φ admits second order approximation around a point c with respect to μ provided that $f(t)$ is μ -derivable at c and that μ is nonlacunar at c (see [2] §5). Written at full length the approximation reads, with the same meaning for $\xi(J)$ as in the above,

$$\Phi(J) = f(c)\mu(J) + 2^{-1}(\mu)f'(c)\xi(J)\mu^2(J) + o[\mu^2(J)].$$

REMARK. Although irrelevant to the purpose of the present note, it would be interesting to investigate what will become of the above assertion when continuity of μ is dropped. Again, we shall only make use of a very special case of this lemma later (§5); i.e. the case in which μ is nowhere lacunar and consequently the following proof can be much simplified.

PROOF. Let us keep the point c fixed. Since μ is continuous we can choose a finite point-function $s(t)$ such that $s(c)=0$ and that $s(I) = \mu(I)$ for all closed intervals I . Then $s(t)$ is continuous and non-decreasing, and the theorem on p. 100 of Saks [7] shows that $\mu(E) = |s[E]|$ for every bounded Borel set E . The image $s[\mathbf{R}]$ of the real line is evidently an interval, for which we shall write H . An interval (finite or infinite) will be called *lacunar* for the nonce, if it is the inverse image under the mapping s of some point of H . Clearly $s^{-1}(u)$ is either a one-point set or else a lacunar interval, for each point u of H . Without loss of generality we may suppose that all lacunar intervals are finite and that, therefore, H is an endless interval. The letter J will stand as above for any closed interval of which c is an endpoint. Then the image $s[J]$, which we shall denote by K for brevity, must always be a closed interval which has zero for one of its endpoints.

It is readily seen that, in order to prove the lemma, we may assume $f(t)$ constant on each lacunar interval. We then can define on H a point-function $F(u)$ uniquely, setting $F(u)=f(t)$ for $u \in H$, where t is any point of \mathbf{R} satisfying $s(t)=u$. This function F fulfils the evident relation $F'(0)=(\mu)f'(c)$. Now, according to a famous theorem belonging to the theory of analytic sets (*vide* Kuratowski [6], p. 251), every biunique image of a Borel set of real numbers under a continuous mapping of \mathbf{R} into itself is likewise a Borel set. It follows easily that $F(u)$ is not only B-measurable on H , but also integrable over each closed interval A in H with respect to Lebesgue measure. The integral which thus arises will be written $\Psi(A)$ for short, and every $\Psi(K)$ is at once found to coincide with $\Phi(J)$ on account of the relation $K=s[J]$ (see above).

This being so, let us confine ourselves only to such J that $\xi(J)=1$,

as we plainly may by symmetry. In virtue of the results proved already the formula of our lemma then transforms at once into

$$\Psi(K) = F(0)|K| + 2^{-1}F'(0)|K|^2 + o(|K|^2).$$

But this is a direct consequence of the evident relation

$$F(u) = F(0) + F'(0)u + o(u) \quad (u \in H, u \rightarrow 0).$$

5. **Second theorem.** We shall resume the consideration of the curve φ of §2, retaining the notations $S(I), \Omega(I), S_*, \Omega_*, \rho(t)$. Given any point u of R , let J_1 and J_2 signify, throughout this section, an arbitrary pair of closed intervals of which u is the right-hand and the left-hand endpoint respectively. Writing $J = J_1 \cup J_2$ we henceforward suppose J so short that neither of the increments $\varphi(J_1)$ and $\varphi(J_2)$ vanishes. This is feasible on account of the proposition of [1] §80. We regard the angle $\varphi(J_1) \diamond \varphi(J_2)$ between the two increments as a function of the interval J and denote it by $G(J)$.

THEOREM. *Let M denote the set of all the points u at which the curvature $\rho(u)$ of the curve φ and the interior S -derivative $(S)G'(u)$ of the function $G(J)$ exist both (as finite or infinite values) and fulfil the relation $\rho(u) = 2(S)G'(u)$. Then we have $S_*(R - M) = 0$.*

REMARK. At present we do not know whether the counterpart relation $\Omega_*(R - M) = 0$ for the measure-bend is also true. Neither is it clear whether M is a Borel set.

PROOF. Let $\gamma(t)$ be a direction-curve for φ as in §2, with Λ and Λ_* for the spheric length and spheric measure-length determined by γ respectively. Then $\gamma(t)$ is B-measurable, i.e. its coordinate-functions are so, since it is locally rectifiable in accordance with the lemma of §2. Further $\Omega_* = \Lambda_*$ identically by the same lemma. Noting that S and S_* coincide for closed intervals in virtue of continuity of φ , we deduce from Tonelli's theorem of [3] §4 that the S -derivative $(S)\varphi'(t)$ exists and equals $\gamma(t)$ for S_* -almost every point t . On the other hand, by change of parameter from t to a length-function $s(t)$ for the curve φ (so that $S_* = s^*$ identically), we infer easily from the first half of Theorem (7.8) on p. 121 of Saks [7] that, for every closed interval Q , the increment $\varphi(Q)$ equals the S_* -integral of $(S)\varphi'(t)$ over Q (cf. also the theorem on p. 100 of Saks [7]). It follows that $\varphi(Q)$ is the S_* -integral of $\gamma(t)$ over Q .

Consider now the Borel set D of all the points t at which $\gamma(t)$ is S -derivable. Given any point u of D , the lemma of the preceding section shows, in view of what has already been proved, that

$$\varphi(J_i)/S(J_i) = \gamma(u) + (-1)^i 2^{-1} S(J_i) \cdot (S)\gamma'(u) + o[S(J_i)] \quad (i=1, 2)$$

as the interval $J = J_1 \cup J_2$ tends to the point u . From this we derive readily

$$G(J) = 2^{-1} S(J) |(S)\gamma'(u)| + o[S(J)],$$

since the two vectors $\gamma(u)$ and $(S)\gamma'(u)$ are orthogonal. The last re-

lation plainly implies that $(S)G'(u)$ exists and equals $2^{-1}|(S)\gamma'(u)|$.

Needless to say, the curve $\gamma(t)$ is continuous at every point of D . Hence its measure-length coincides with Λ_* for any Borel set X (bounded or not) in D on account of the lemma of [5] §5. It follows from the Supplement of [3] §4 and the decomposition theorem of [2] §8 that

$$\Lambda_*(X) = \int_X |(S)\gamma'(t)| dS_*(t) = \Lambda_*(AX) + \int_X (S)\Lambda'_*(t) dS_*(t)$$

for such X , where A denotes the Borel set of the points t of D for which $(S)\Lambda'_*(t)$ becomes $+\infty$. Replacing X by A we readily get $\Lambda_*(A) = 0$ since $S_*(A) = 0$ by Lebesgue's theorem of [2] §4. Consequently the above two integrals over X must coincide. It is then easy to see that $S_*(D - D_0) = 0$, where D_0 stands for the Borel set of the points t of D at which $(S)\Lambda'_*(t)$ exists and equals $|(S)\gamma'(t)|$.

Let us turn now to the set M of the assertion. In view of the identity $\Omega_* = \Lambda_*$, remarked at the beginning, and of the equation $|(S)\gamma'(u)| = 2(S)G'(u)$, obtained above for $u \in D$, we find by the theorem of [4] §7 that M contains D_0 , so that $D - M \subset D - D_0$. It follows that $S_*(D - M) = 0$. But $S_*(R - D) = 0$ in virtue of the lemma of [3] §3. We thus derive finally

$$S_*(R - M) \leq S_*(R - D) + S_*(D - M) = 0,$$

which completes the proof of the theorem.

References

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