75. On the Blackwell Theorem in Operator Algebras

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(Comm. by K. KUNUGI, M.J.A., June 12, 1961)

1. D. Blackwell [1] established, among others, the following theorem: *If $f_1, f_2, \cdots, f_n$ are integrable with respect to a non-atomic probability measure u on a measurable space $(X, \mathfrak{A})$, then there is a sigma-subfield $\mathfrak{B}$ on which u is non-atomic and*

$$(1) \qquad \int_D f_i(x)\, du(x) = u(D) \int_X f_i(x)\, du(x), \qquad i=1, 2, \cdots, n,$$

for every $D \in \mathfrak{B}$. It is important in the theory of statistics that the theorem of Blackwell implies the well-known Lyapnov convexity theorem on the ranges of vector measures.[3)]

Since the theory of von Neumann algebras of finite type is recognized as a non-commutative extension of the probability theory,[4)] and since (1) is equivalent to

$$(2) \qquad E[f_i \mid \mathfrak{B}] = E[f_i], \qquad i=1, 2, \cdots, n,$$

where $E[g \mid \mathfrak{B}]$ (respectively $E[g]$) is the conditional expectation of g conditioned by $\mathfrak{B}$ (respectively the expectation of g), it may be observed with some interests that the Blackwell theorem has a non-commutative extension with the same words in the following

THEOREM. *If A is a continuous finite von Neumann algebra with a faithful normal trace τ, and if $a_1, a_2, \cdots, a_n$ are hermitean elements of A with*

$$(3) \qquad \tau(a_i) = 0, \qquad i=1, 2, \cdots, n,$$

then there is a continuous subalgebra B such as

$$(4) \qquad a_i^{\epsilon} = 0, \qquad i=1, 2, \cdots, n,$$

where a^{ϵ} is the conditional expectation of a conditioned by B in the sense of [5].

If A is abelian, the theorem becomes the theorem of Blackwell in the above. Moreover, the proof of the theorem can be carried out in the same method of Blackwell with a few minor modifications, as will be seen in the below.

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3) Lyapnov's theorem and the allied topics are discussed in a recent exposition [3] of Dubins and Spanier, where Lyapnov's theorem is given a proof without appealing the theorem of Blackwell.

4) The terminology of J. Dixmier [2] will be used without any explanation. A list of non-commutative generalizations of theorems on additive set functions will be found in [4].

2. As Blackwell did, the proof of the theorem is reduced to the following simplest case:

LEMMA 1. *For A stated in the theorem, if a is an hermitean element of A with $\tau(a)=0$, then there is a continuous subalgebra B satisfying*

$$a^{\epsilon}=0. \tag{4'}$$

It will be shown at first that Lemma 1 implies the theorem. By Lemma 1, for the given A and $a=a_1$, there is a continuous subalgebra B_1 satisfying (4'). For B_1 and $E[a_2|B_1]$,[5] Lemma 1 also guarantees that a continuous subalgebra B_2 satisfies $E[E[a_2|B_1]|B_2]=0$. Since B_2 is a subalgebra of B_1, a property of the conditional expectation implies

$$E[a_2|B_2]=E[E[a_2|B_1]|B_2]=0,$$

as required. Inductively, there is a sequence of subalgebras $B_1\geqq B_2\geqq\cdots\geqq B_n$, and $B=B_n$ has the required properties by the construction, which proves the theorem.

To prove Lemma 1, it requires that the following variant of the Bisection Theorem holds for finite von Neumann algebras:

LEMMA 2. *If A is continuous finite, if a is an hermitean operator of A satisfying $\tau(a)=0$, and if e is a projection of A with $\tau(ae)=0$, then there is a projection $p\leqq e$ such that*

$$\tau(ap)=0, \tag{5}$$

and

$$\tau(p)=\frac{1}{2}\tau(e). \tag{6}$$

It will be shown here that Lemma 2 implies Lemma 1. Putting $e=1$, Lemma 2 insures that there is a pair of projections p_1 and p_2 satisfying (5) such that $p_1+p_2=1$ and $\tau(p_1)=\tau(p_2)=\frac{1}{2}$. Again, putting $e=p_i$ (for $i=1,2$), there is a set of mutually orthogonal projections p_{21}, p_{22}, p_{23}, and p_{24} satisfying (5) and $\tau(p_{2j})=\frac{1}{4}$ for $j=1,2,3,4$. Inductively, one has sets of mutually orthogonal projections $\{p_{ij}|1\leqq j\leqq 2^i\}$ for $i=1,2,\cdots$ satisfying (5) and $\tau(p_{ij})=(\frac{1}{2})^i$ (for $1\leqq j\leqq 2^i$). If C_n is the von Neumann subalgebra generated by $\{p_{nj}|1\leqq j\leqq 2^n\}$, then $C_1\leqq C_2\leqq\cdots$. It is obvious that C_n satisfies $E[a|C_n]=0$ since every projection of C_n satisfies (5). Let C_∞ be the von Neumann subalgebra generated by $\{C_n\}$, then

$$\{E[a|C_n]\,|\,n=1,2,\cdots,\infty\} \tag{7}$$

is a martingale in the sense of [6]. Since $E[a|C_n]=0$ for $n=1,2,\cdots$ and since (7) is a simple martingale, $E[a|C_\infty]=0$ by the martingale

5) For printing convenience, the notation of probabilists is used here. For properties of the conditional expectation, cf. [5] and [6].

theorem.[6] Putting $B=C_\infty$, B becomes the subalgebra satisfying the required properties of the lemma, since it is obvious by the construction that B is non-atomic.

REMARK 1. It will be shown here that B can be chosen *maximal* among such subalgebras in Lemma 1. Let Φ be the collection of all continuous von Neumann subalgebras satisfying (4′). Then Φ is a non-void inductively ordered set by inclusion according to Lemma 2. Hence, there is a maximal continuous von Neumann subalgebra which satisfies (4′).

REMARK 2. It is also possible to require, with a few modifications in the above proof, that the von Neumann subalgebra B is contained in the commutor $(a)'$ of a, i.e. each element of B commutes with a.

3. It remains to show that the usual Bisection Theorem for measure spaces implies the general Lemma 2.[7] Let B be an abelian von Neumann subalgebra containing the given e. Then B can be thought of the multiplication algebra on the spectrum Ω of B with the measure τ. If $\sigma(x)=\tau(ax)$, then σ defines a measure on Ω which is absolutely continuous with respect to τ. Hence the usual Bisection Theorem implies the existence of a projection p which satisfies the requirements of Lemma 2.

4. In the remainder, it will be shown briefly that Lemma 1 has an another proof without appealing Lemma 2.

At first, using the Jordan decomposition $a=a'-a''$, one can define two positive linear functionals $\rho'(x)=\tau(a'x)$ and $\rho''(x)=\tau(a''x)$ with their supports e' and e'' respectively. Under these definitions, it is not hard to see that the following fact holds: *For any non-zero projection $p'<e'$, there is a non-zero projection $p''<e''$ such as $\rho'(p')=\rho''(p'')$. Hence, putting $p=p'+p''$, there is a projection p such that p satisfies* (5) *and* $0<p<1$.

Let Ψ be the collection of all von Neumann subalgebras satisfying (4′). Then Ψ is an inductively ordered set by inclusion and non-void by the above fact. Hence there is a maximal von Neumann subalgebra C in Ψ. It is sufficient to show that *C is continuous.*

If C contains an atom p, then the above argument also guarantees that there is a non-zero projection $q<p$ such as $\tau(aq)=0$. Since p is an atom of C, q is clearly excluded by C, whence the von Neumann subalgebra generated by C and p contains C properly and belongs to Ψ, which contradicts the maximality of C.

6) A martingale is called an M-net in [6]. The martingale theorem of Doob is extended for operator algebras in [6, Theorem 2].

7) It is noteworthy that a similar argument admits to derive Lemma 1 from the usual Blackwell Theorem.

References

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