74. On the Dimension of an Orbit-space

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Let G be a locally compact transformation group satisfying the second axiom of countability and acting on a locally compact Hausdorff space M, and H be a compact invariant subgroup of G. Then in a natural way the set of all orbits under H becomes a locally compact Hausdorff space, which is called "the orbit-space of M under H" and denoted by $D(M;H)$, and the factor group $G^*=G/H$ acts on $D(M;H)$ as a transformation group (cf. [4], p. 61). In this note we prove that

$$\dim G(x)=\dim H(x)+\dim D(G(x);H) \quad \text{for } x\in M. \tag{A}$$

This is a generalization of a result obtained by Montgomery and Zippin ([5], p. 783, cf. Corollary of the present note). If $G(x)$ is finite dimensional, then $D(G(x);H)$ is locally the topological product of a Euclidean cube by a zero dimensional set closed in $D(G(x);H)$ (cf. Karube [3]); so that the equation (A) gives us the almost complete knowledge about the local topology of such an orbit-space as the above.

We now prove the equation (A).

1) *Let G be finite dimensional.* Let p be the natural projection of M onto $D(M;H)$, and $\tilde{x}$ the image of the point x under p. Let π be the natural mapping of G onto G^*, F^* the group of all elements of G^* leaving the point $\tilde{x}$ fixed, and F the complete inverse image of F^* under π. It is easy to see that $F(x)=H(x)$ and $G_x=F_x$ where G_x and F_x are stability subgroups of the point x. By the theorems of Yamanoshita [6] we have

$$\dim G=\dim F+\dim G/F,$$
$$\dim G=\dim G(x)+\dim G_x,$$
$$\dim F=\dim F(x)+\dim F_x=\dim H(x)+\dim G_x,$$
$$\dim G/F=\dim G^*/F^*=\dim G^*(\tilde{x})=\dim D(G(x);H).$$

Since G_x is finite dimensional, we have (A).

2) *Let $G(x)$ be finite dimensional.* There exists an open subgroup G' of G such that G'/G_0 is compact where G_0 is the identity component of G. Since $G'(x)$ is finite dimensional, G' is effectively finite dimensional on $G'(x)$. In fact, there must be a connected compact invariant subgroup K' of G' which is idle on $G'(x)$ and such that G'/K' is finite dimensional (cf. [3]). Let G_1' be the factor group G'/K', ρ the natural mapping of G' onto G_1', H' the intersection of H and G', and H_1' the image of H' under ρ. Since G_1' is finite dimensional we have

$$\dim G_1'(x)=\dim H_1'(x)+\dim D(G_1'(x); H_1'),$$

i.e. $$\dim G'(x)=\dim H'(x)+\dim D(G'(x); H').$$

It is easily seen that $\dim G'(x)=\dim G(x)$, because $\dim G_0(x)=\dim G(x)$ by a theorem of Yamanoshita [6]. In a similar way we have

$$\dim H'(x)=\dim H(x) \text{ and } \dim D(G'(x); H')=\dim D(G(x); H).$$

Hence we have (A).

3) *Let $G(x)$ be infinite dimensional.* We can suppose without loss of generality that $D(G(x); H)$ is finite dimensional. Let W be a compact neighborhood of x in M, and put $U=G(x)\cap W$. Let V be the image of U under p and p' be the contraction of p on U, then we see as follows that p' is a closed mapping of U onto V. Let C be any closed set in U and $\tilde{C}$ the image of C under p, then

$$p'^{-1}(V-\tilde{C})=U-(H(C)\cap U).$$

Since $H(C)$ is closed in $G(x)$ ([1], p. 37), $p'^{-1}(V-\tilde{C})$ is open in U. Hence $\tilde{C}$ is closed in V. Now if $H(x)$ were finite dimensional, there would be an integer m such that the dimension of $p'^{-1}(\tilde{y})$ does not exceed m for any point $\tilde{y}$ of V, and so

$$\dim U\leqq m+\dim V \quad \text{(Hurewicz and Wallman [2], p. 92)}$$

i.e. $$\dim G(x)\leqq m+\dim D(G(x); H).$$

This contradicts our hypotheses on dimensions of $G(x)$ and $D(G(x); H)$. Hence $H(x)$ is infinite dimensional.

Consequently we have the following theorem.

Theorem. *Let G be a locally compact transformation group satisfying the second axiom of countability and acting on a locally compact Hausdorff space M, and H be a compact invariant subgroup of G. Let $D(G(x); H)$ be the orbit-space of $G(x)$ under H, then*

$$\dim G(x)=\dim H(x)+\dim D(G(x); H)$$

for any point x of M.

Corollary. *Let G, H and M be the same as the above respectively. If G acts transitively on a finite dimensional connected space M and H is not transitive on M, then* $\dim H(x)$ *is less than* $\dim M$.

Proof. $D(M; H)$ is connected and not a single point; so that $D(M; H)$ is positive dimensional.

Example. Let M be a Euclidean plane, T the group of all translations in M, H the group of all rotations around a fixed point p in M, and G the group generated by T and H. Then the relation (A) does not hold. In this case H is *not invariant* subgroup of G.

References

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