73. A Generalization of the Heinz Inequality

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The object of the present note is to generalize the Heinz inequality for selfadjoint operators to a wider class of accretive operators.

A linear operator A in a Hilbert space is said to be *accretive*[1] if $\mathrm{Re}(Au, u)\geq 0$ for all $u\in\mathfrak{D}[A]$ ($\mathfrak{D}[A]$ is the domain of A). If A is closed and maximal accretive, then A is densely defined, and the fractional powers A^h are defined for $0\leq h\leq 1$ and are again closed and maximal accretive.[2]

Our main result is given by

Theorem 1. *Let A, B be closed, maximal accretive operators in Hilbert spaces $\mathfrak{H}$, $\mathfrak{H}'$, respectively, and let T be a bounded linear operator*[3] *from $\mathfrak{H}$ to $\mathfrak{H}'$. If $T\mathfrak{D}[A]\subset\mathfrak{D}[B]$ and*

$$(1)\qquad \|BTu\|\leq M\|Au\|,\quad u\in\mathfrak{D}[A],$$

with a constant M, then we have $T\mathfrak{D}[A^h]\subset\mathfrak{D}[B^h]$ and

$$(2)\qquad \|B^hTu\|\leq e^{ch(1-h)}M^hN^{1-h}\|A^hu\|,\quad u\in\mathfrak{D}[A^h],$$

where $N=\|T\|$, $0\leq h\leq 1$ and c is an absolute constant. We can take $c=0$ if A, B are selfadjoint and nonnegative. In general we can take $c=\pi^2/2$, but we do not know whether this is the optimal value.

Remark. The value of c can be improved if A, B are themselves fractional powers of accretive operators. Suppose that there are closed, maximal accretive operators A_1, B_1 in $\mathfrak{H}$, $\mathfrak{H}'$, respectively, such that $A=A_1^s$, $B=B_1^t$ for some s, t, $0<s\leq 1$, $0<t\leq 1$. Then we can set $c=\pi^2(s^2+t^2)/4$. (The proof is not essentially different from the proof of Theorem 1 given below.) If, for example, A is nonnegative selfadjoint, we can make $s\to 0$ and set $c=\pi^2t^2/4$.

Corollary. *If A, B are closed, maximal accretive operators in $\mathfrak{H}$ such that $\mathfrak{D}[A]\subset\mathfrak{D}[B]$ and $\|Bu\|\leq\|Au\|$ for $u\in\mathfrak{D}[A]$, then $\mathfrak{D}[A^h]\subset\mathfrak{D}[B^h]$ and $\|B^hu\|\leq e^{ch(1-h)}\|A^hu\|$ for $u\in\mathfrak{D}[A^h]$, $0\leq h\leq 1$.*

Theorem 1 is equivalent to

Theorem 2. *Let A, B be as in Theorem 1, and let Q be a densely*

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1) Then $-A$ is said to be *dissipative*. For the term "accretive", see K. O. Friedrichs: Symmetric positive linear differential equations, Comm. Pure Appl. Math., **11**, 333-418 (1958).

2) See T. Kato: Fractional powers of dissipative operators, J. Math. Soc. Japan, **13** (1961), in press. This paper will be quoted as (F) in the following.

3) A bounded linear operator is assumed to be defined everywhere in the domain space, unless otherwise stated explicitly.

defined, closed linear operator from $\mathfrak{H}$ *to* $\mathfrak{H}'$ *such that* $\mathfrak{D}[A]\subset\mathfrak{D}[Q]$, $\mathfrak{D}[B]\subset\mathfrak{D}[Q^*]$ *and*

(3) $\|Qu\|\leq\|Au\|, \quad u\in\mathfrak{D}[A]; \quad \|Q^*v\|\leq\|Bv\|, \quad v\in\mathfrak{D}[B].$

Then we have for $0\leq h\leq 1$

(4) $|(Qu, v)|\leq e^{ch(1-h)}\|A^h u\|\,\|B^{1-h}v\|, \quad u\in\mathfrak{D}[A], \quad v\in\mathfrak{D}[B].$

In the selfadjoint case ($c=0$), these results are known as the Heinz inequality.[4] Recently, Krasnosel'skii and Sobolevskii[5] considered the generalization of the Heinz inequality to non-selfadjoint operators in Hilbert and Banach spaces. But their results are different from (2) in that the A^h on the right is replaced by A^k with a $k>h$ (with the numerical factor depending on h and k) and similarly for (4).

We first prove Theorem 1 in the following weakened form.

Theorem 3. *Let* A, B *be bounded accretive operators in* $\mathfrak{H}$, $\mathfrak{H}'$, *respectively, and let* T *be a bounded linear operator from* $\mathfrak{H}$ *to* $\mathfrak{H}'$. *Then*

(5) $\|B^h T A^h\|\leq e^{ch(1-h)}\|T\|^{1-h}\|BTA\|^h, \quad 0\leq h\leq 1.$

To prove this, we need a lemma which generalizes a previous result of the author.

Lemma. *Let* A *be a closed, maximal accretive operator in* $\mathfrak{H}$. *Then there is, for each* α *with* $0<\alpha<1/2$, *a bounded linear operator* U_α *in* $\mathfrak{H}$ *such that*

(6) $A^{*\alpha}=U_\alpha A^\alpha, \quad \|U_\alpha\|\leq c_\alpha,$

where c_α *is a constant depending only on* α *and*

(7) $\limsup_{\alpha\to 0}(c_\alpha-1)/\alpha^2=c\leq\pi^2/2.$

Proof of Lemma. The existence of a U_α with the property (6) follows from the result $\|A^{*\alpha}u\|\leq c_\alpha\|A^\alpha u\|$, $u\in\mathfrak{D}[A^\alpha]=\mathfrak{D}[A^{*\alpha}]$, which is proved in (F) (see Theorem 1.1 of (F)). However, the constant $c_\alpha=\tan[(1+2\alpha)/4\pi]$ deduced in (F) does not satisfy (7). Let us now improve this c_α. We first assume that A is bounded and $\mathrm{Re}(Au, u)\geq a\|u\|^2$, $a>0$, and note, following the notation of (F), that $A^\alpha=H_\alpha+iK_\alpha$, $A^\alpha H_\alpha^{-1}=1+iK_\alpha H_\alpha^{-1}$. Hence

(8) $\|A^\alpha H_\alpha^{-1}u\|^2=\|u\|^2+\|K_\alpha H_\alpha^{-1}u\|^2+i((K_\alpha H_\alpha^{-1}-H_\alpha^{-1}K_\alpha)u, u).$

Here we have

(9) $\|K_\alpha H_\alpha^{-1}-H_\alpha^{-1}K_\alpha\|\leq 2\tan^2(\pi\alpha/2), \qquad 0\leq\alpha\leq 1/2.$

To see this, we consider $X_\alpha=K_\alpha H_\alpha^{-1}-H_\alpha^{-1}K_\alpha$ for complex α. We know

4) E. Heinz: Beiträge zur Störungstheorie der Spektralzerlegung, Math. Ann., **123**, 415–438 (1951); T. Kato: Notes on some inequalities for linear operators, Math. Ann., **125**, 208–212 (1952); J. Dixmier: Sur une inégalité de E. Heinz, Math. Ann., **126**, 75–78 (1953); E. Heinz: On an inequality for linear operators in a Hilbert space, Report of an International Conference on Operator Theory and Group Representations, Arden House, Harriman, N. Y., 27–29 (1955); H. O. Cordes: A matrix inequality, Proc. Amer. Math. Soc., **11**, 206–210 (1960).

5) M. A. Krasnosel'skii and P. E. Sobolevskii: Fractional powers of operators acting in Banach spaces (in Russian), Doklady Acad. Nauk, **129**, 499–502 (1959).

by (F) that X_α is holomorphic and $\| K_\alpha H_\alpha^{-1} \| \leq 1$, $\| H_\alpha^{-1} K_\alpha \| = \| (K_{\bar\alpha} H_{\bar\alpha}^{-1})^* \| \leq 1$ for $|\operatorname{Re}\alpha| \leq 1/2$. Hence $\| X_\alpha \| \leq 2$ for $|\operatorname{Re}\alpha| \leq 1/2$. But it is easily seen that X_α has a double zero at $\alpha=0$ (note that $K_0=0$, $H_0=1$). Therefore $X_\alpha/\tan^2(\pi\alpha/2)$ is holomorphic for $|\operatorname{Re}\alpha| \leq 1/2$, and (9) follows by the maximum principle as in (F).

Thus we have from (8)

$$[1-2\tan^2(\pi\alpha/2)](\| u \|^2 + \| K_\alpha H_\alpha^{-1} u \|^2) \leq \| A^\alpha H_\alpha^{-1} u \|^2 \leq [1+2\tan^2(\pi\alpha/2)](\| u \|^2 + \| K_\alpha H_\alpha^{-1} u \|^2).$$

Since the same inequalities hold for A replaced by A^*, we have for all $v \in \mathfrak{H}$

(10) $$\| A^{*\alpha} v \| \leq [1+2\tan^2(\pi\alpha/2)]^{1/2}[1-2\tan^2(\pi\alpha/2)]^{-1/2} \| A^\alpha v \|,$$

at least for $0 \leq \alpha \leq 1/3$. As in (F), (10) can then be extended to the general case by a limiting procedure. (10) implies the existence of a U_α with the property (6), with $c_\alpha = [1+2\tan^2(\pi\alpha/2)]^{1/2}[1-2\tan^2(\pi\alpha/2)]^{-1/2}$ for $0 \leq \alpha \leq 1/3$ and $c_\alpha = \tan[(1+2\alpha)/4\pi]$ for $1/3 < \alpha < 1/2$. This c_α satisfies (7) with $c = \pi^2/2$.

Proof of Theorem 3. Let $d>0$, $h-d \geq 0$, $h+d \leq 1$. We have

(11) $$\| B^h T A^h \|^2 \leq \| A^{*h} T^* B^{*h} B^h T A^h \| \leq \| A^{*h-d} T^* B^{*h-d} B^{*d} B^h T A^h A^{*d} \|,$$

where we have used the facts that $\| R \|^2 = \| R^* R \|$ in general and $\| RS \| \leq \| SR \|$ if RS is symmetric.[6] If $d<1/2$, we have $B^{*d} = V_d B^d$ with $\| V_d \| \leq c_d$ by Lemma. Similarly, we have $A^d = W_d A^{*d}$ or $A^{*d} = A^d W_d^*$ with $\| W_d \| \leq c_d$. Hence (11) gives

$$\| B^h T A^h \|^2 \leq \| A^{*h-d} T^* B^{*h-d} \| \, \| V_d B^{h+d} T A^{h+d} W_d^* \| \leq c_d^2 \| B^{h-d} T A^{h-d} \| \, \| B^{h+d} T A^{h+d} \|.$$

Setting $f(h) = \log \| B^h T A^h \|$, we have $2f(h) - f(h-d) - f(h+d) \leq 2\log c_d$. Dividing both sides by d^2 and letting $d \to 0$, we obtain by (7)

$$\limsup_{d\to 0} d^{-2}[2f(h) - f(h-d) - f(h+d)] \leq 2c.$$

It follows that $f(h) \leq (1-h)f(+0) + hf(1) + ch(1-h)$, which is equivalent to (5).

Proof of Theorem 1. i) If A, B and A^{-1} are bounded, (1) implies that $\| BTA^{-1} \| \leq M$. Hence we have, by Theorem 3, $\| B^h T A^{-h} \| \leq e^{ch(1-h)} M^h N^{1-h}$, which implies (2). ii) If A, B are bounded, (1) implies that $\| BTu \| \leq M \| (A+\varepsilon)u \|$ for $\varepsilon>0$. Since $(A+\varepsilon)^{-1}$ is bounded, by i) we obtain (2) with A^h replaced by $(A+\varepsilon)^h$. Then we get (2) by letting $\varepsilon \to 0$, for $(A+\varepsilon)^h \to A^h$ strongly (see (F)). iii) Assume that A, B are not necessarily bounded but A^{-1}, B^{-1} are bounded. Then it is easily seen[7] that (1) is equivalent to $\| A^{*-1} T^* v \| \leq M \| B^{*-1} v \|$ for all $v \in \mathfrak{H}'$. Thus we have $\| A^{*-h} T^* v \| \leq e^{ch(1-h)} M^h N^{1-h} \| B^{*-h} v \|$ by ii),

6) RS and SR have the same spectral radius r. The symmetry of RS implies $\| RS \| = r$, whereas $\| SR \| \geq r$. Cf. the paper by Cordes cited in 4).

7) Cf. the paper by Kato cited in 4).

and this is again equivalent to (2). iv) In the general case, (1) implies that

$$\|(B+\varepsilon^2)Tu\| \leq M\|Au\| + \varepsilon^2 N\|u\| \leq (M^2+\varepsilon^2N^2)^{1/2}(\|Au\|^2+\varepsilon^2\|u\|^2)^{1/2}$$
$$\leq (M^2+\varepsilon^2N^2)^{1/2}\|(A+\varepsilon)u\| \qquad \text{for} \quad \varepsilon>0.$$

Thus by iii) we have (2) with A, B, M replaced by $A+\varepsilon$, $B+\varepsilon^2$, $(M^2+\varepsilon^2N^2)^{1/2}$ respectively. Then (2) follows by letting $\varepsilon\to 0$; note that $\mathfrak{D}[(A+\varepsilon)^h]=\mathfrak{D}[A^h]$ and $(A+\varepsilon)^h u\to A^h u$ for $u\in\mathfrak{D}[A^h]$, see (F).

Proof of Theorem 2. The second inequality of (3) implies that there is a bounded linear operator T^* from $\mathfrak{H}'$ to $\mathfrak{H}$ such that $T^*B \subset Q^*$, $\|T^*\|\leq 1$. Then it follows easily that $Q\subset B^*T$. Thus the first inequality of (3) implies that $\|B^*Tu\|\leq\|Au\|$ and we see by Theorem 1 that $\|B^{*h}Tu\|\leq e^{ch(1-h)}\|A^h u\|$. Hence $|(Qu,v)|=|(B^*Tu,v)| =|(B^{*h}Tu, B^{1-h}v)|\leq e^{ch(1-h)}\|A^h u\|\,\|B^{1-h}v\|$. We can also deduce Theorem 1 from Theorem 2, but the proof may be omitted.

As an application of Theorem 1 (Corollary), let us prove

Theorem 4. *Let A be a closed, maximal accretive operator in $\mathfrak{H}$. Then the selfadjoint operators $(A^*A)^h$ and $(AA^*)^h$ are comparable for $0\leq h<1/4$ (that is, $\mathfrak{D}[(A^*A)^h]=\mathfrak{D}[(AA^*)^h]$ and the values $\|(A^*A)^h u\|/\|(AA^*)^h u\|$ are bounded by positive constants from above and from below).*

Proof. It is well known that A and $(A^*A)^{1/2}$ are comparable. Hence A^h and $(A^*A)^{h/2}$ are comparable for $0\leq h\leq 1$ by Corollary to Theorem 1. Similarly A^{*h} and $(AA^*)^{h/2}$ are comparable. But A^h and A^{*h} are comparable for $0\leq h<1/2$ by Lemma (cf. also (F)). Hence follows Theorem 4.

Example. Let $\mathfrak{H}=L^2(0,\infty)$ and $A=d/dx$ with the boundary condition $u(0)=0$. A is maximal accretive and $A^*=-d/dx$ with no boundary condition. Thus $A^*A=-d^2/dx^2$ with the Dirichlet boundary condition $u(0)=0$, whereas $AA^*=-d^2/dx^2$ with the Neumann boundary condition $u'(0)=0$. Obviously these two selfadjoint operators have different domains, but Theorem 4 shows that their h-th powers are comparable if $0\leq h<1/4$. A similar result can be proved for second order elliptic differential operators in n variables, although the proof given here is not applicable.