72. Inverse Images of Closed Mappings. II

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In the following, we deal chiefly with the case when the inverse images of closed continuous mappings become normal.

Theorem 5. *Let $f(X)=Y$ be a closed continuous mapping of a topological space X onto a paracompact Hausdorff space Y. Then X is normal if and only if, for each point y of Y, any two disjoint closed subsets A, B of the inverse image $f^{-1}(y)$ can be separated by open sets of X, that is, there exist open sets G, H of X such that $G\supset A$, $H\supset B$ and $G\frown H=\phi$.*

Proof. The "only if" part is obvious. So that we shall prove the "if" part. Let A and B be two disjoint closed sets of X and let G be an open set of X. Then we can see that the set $\{y\,|\,f^{-1}(y)\frown A\subset G\}$ is an open set of Y. In fact, let y_0 be any point such that $f^{-1}(y_0)\frown A\subset G$ and let $V=Y-f(A\frown(X-G))$. Then, since f is a closed continuous mapping, V is an open set of Y and $y_0\in V$, $f^{-1}(V)\frown A\frown(X-G)=\phi$. Hence $f^{-1}(V)\frown A\subset G$. Therefore the set $\{y\,|\,f^{-1}(y)\frown A\subset G\}$ is an open set of Y. Now let $U_G=\{y\,|\,f^{-1}(y)\frown A\subset G, f^{-1}(y)\frown B\subset X-\bar{G}\}$, then U_G is an open set of Y. For any point y_0 of Y, $f^{-1}(y_0)\frown A$ and $f^{-1}(y_0)\frown B$ are disjoint closed sets of $f^{-1}(y_0)$. By assumption, there exist two open sets G_0, H_0 of X such that $f^{-1}(y_0)\frown A\subset G_0$, $f^{-1}(y_0)\frown B\subset H_0$ and $G_0\frown H_0=\phi$. Since $\bar{G}_0\frown H_0=\phi$, we get $H_0\subset X-\bar{G}_0$. Hence $y_0\in U_{G_0}$. Then we can see that the family of open sets $\{U_G\,|\,G$ ranges over all open sets of $X\}$ is an open covering of Y. Since Y is paracompact Hausdorff space, there exists a locally finite open covering $\{V_G\,|\,G\in\mathfrak{G}\}$ where $\mathfrak{G}$ is a family of open sets of X such that $\overline{V_G}\subset U_G$ for every $G\in\mathfrak{G}$. Let $H=\bigcup_{G\in\mathfrak{G}}(f^{-1}(V_G)\frown G)$, then H is an open set of X and $\{f^{-1}(V_G)\frown G\,|\,G\in\mathfrak{G}\}$ is locally finite. Hence $\bar{H}=\bigcup_{G\in\mathfrak{G}}(\overline{f^{-1}(V_G)\frown G})\subset\bigcup_{G\in\mathfrak{G}}(f^{-1}(\overline{V_G})\frown\bar{G})$. On the other hand, since $f^{-1}(V_G)\frown A\subset f^{-1}(U_G)\frown A\subset G$, we get $f^{-1}(V_G)\frown A\subset f^{-1}(V_G)\frown G\subset H$. Since $\{f^{-1}(V_G)\,|\,G\in\mathfrak{G}\}$ covers X, we get $A\subset H$. On the other hand, $f^{-1}(\overline{V_G})\frown B\frown\bar{G}\subset f^{-1}(U_G)\frown B\frown\bar{G}\subset(X-\bar{G})\frown\bar{G}=\phi$. Then $B\frown\bar{H}=\phi$. Hence we have an open set $X-\bar{H}$ which contains B. Therefore A and B are separated by open sets H and $X-\bar{H}$, and so that X is normal. This completes the proof.

Corollary 2.1 (H. Tamano [1]). *If f is a closed continuous mapping of a topological space X onto a paracompact Hausdorff space Y, then X is normal if and only if, for any point y of Y, the inverse image $f^{-1}(y)$ is normal and every bounded continuous function on $f^{-1}(y)$ has a continuous extension over the space X.*

Proof. It is sufficient to show the "if" part. Let A_0 and B_0 be two disjoint closed subsets of $f^{-1}(y_0)$, then there exists a bounded continuous function f_0 defined on $f^{-1}(y_0)$ such that $f_0(x)=0$ for each $x\in A_0$ and $f_0(x)=1$ for each $x\in B_0$. Let f be a continuous extension of f_0 over X and let $G_0=\left\{x\mid f(x)<\frac{1}{2}\right\}$ and $H_0=\left\{x\mid f(x)>\frac{1}{2}\right\}$. Then G_0 and H_0 are open sets of X such that $A_0\subset G_0$, $B_0\subset H_0$ and $G_0\frown H_0=\phi$. Then, by Theorem 5, X is normal. This completes the proof.

Corollary 2.2. *If f is a closed continuous mapping of a Hausdorff space X onto a paracompact Hausdorff space Y such that the inverse image $f^{-1}(y)$ is normal and the boundary $\mathfrak{B}f^{-1}(y)$ is compact for every point y of Y, then X is normal.*

Proof. Let A and B be two disjoint closed subsets of $f^{-1}(y)$, then by the normality of $f^{-1}(y)$, there exist open sets G and H of X such that $A\subset f^{-1}(y)\frown G$, $B\subset f^{-1}(y)\frown H$ and $f^{-1}(y)\frown G\frown H=\phi$.

Since $\mathfrak{B}f^{-1}(y)$ is compact and X is a Hausdorff space, there exist open sets G_0 and H_0 of X such that $\mathfrak{B}f^{-1}(y)\frown A\subset G_0$, $\mathfrak{B}f^{-1}(y)\frown B\subset H_0$ and $G_0\frown H_0=\phi$. Now let $G'=[\mathrm{Int}\, f^{-1}(y)\frown G]\smile[G_0\frown G]$, $H'=[\mathrm{Int}\, f^{-1}(y)\frown H]\smile[H_0\frown H]$, then $A\subset G'$ and $B\subset H'$. Since $(G_0\frown G)\frown[\mathrm{Int}\, f^{-1}(y)\frown H]\subset f^{-1}(y)\frown G\frown H=\phi$, we have $G'\frown H'=\phi$. Hence, by Theorem 5, X is normal. This completes the proof.

Corollary 2.3. *If f is a closed continuous mapping of a regular topological space onto a paracompact Hausdorff space Y such that the inverse image $f^{-1}(y)$ is normal and the boundary $\mathfrak{B}f^{-1}(y)$ has the Lindelöf property for every point y of Y, then X is normal.*

Proof. In the proof of Corollary 2.2, $\mathfrak{B}f^{-1}(y)\frown A$ and $\mathfrak{B}f^{-1}(y)\frown B$ are disjoint closed sets and each of which has the Lindelöf property.

Since X is regular, we can see that there exist open sets G_0 and H_0 of X such that $\mathfrak{B}f^{-1}(y)\frown A\subset G_0$, $\mathfrak{B}f^{-1}(y)\frown B\subset H_0$ and $G_0\frown H_0=\phi$. Hence we can apply the same argument as Corollary 2.2.

Theorem 6. *If f is a closed continuous mapping of a topological space X onto a paracompact Hausdorff space Y, then X is paracompact and normal if and only if the following three conditions are satisfied: for every point y of Y, (a) any two disjoint closed subsets of $f^{-1}(y)$ are separated by open sets of X, (b) $f^{-1}(y)$ is paracompact, (c) for any locally finite open covering $\{U_\alpha\}$ of the boundary $\mathfrak{B}f^{-1}(y)$, there exists a locally finite system $\{V_\alpha\}$ of open sets of X such that $V_\alpha\frown\mathfrak{B}f^{-1}(y)\subset U_\alpha$ for each α* and $\{V_\alpha\}$ covers $\mathfrak{B}f^{-1}(y)$.

Proof. The "only if" part is obvious. We shall prove the "if" part in the following. The normality of X follows from the condition (a) by virtue of Theorem 5. We next prove the paracompactness of X. Let $\{U_\lambda\}$ be an open covering of X, then, since $f^{-1}(y)$ is paracompact, there exists a locally finite open refinement $\{W_\mu\}$ of $\{f^{-1}(y) \frown U_\lambda\}$. Let $W'_\mu = W_\mu \frown \mathrm{Int} f^{-1}(y)$. By the condition (c), there exists a locally finite system $\{W''_\mu\}$ of open sets of X such that $W''_\mu \frown \mathfrak{B} f^{-1}(y) \subset \mathfrak{B} f^{-1}(y) \frown W_\mu$ for each μ and each W''_μ is contained in some U_λ. Then $\{W'_\mu, W''_\mu\}$ is a locally finite system of open sets of X and covers the set $f^{-1}(y)$ and any set of $\{W'_\mu, W''_\mu\}$ is contained in some U_λ.

Let $\{\mathfrak{G}_\alpha \mid \alpha \in A\}$ be the set of all locally finite systems of open sets of X such that, for each α, every set of $\mathfrak{G}_\alpha$ is contained in some U_λ. Then $\{V_\alpha \mid \alpha \in A\}$ where $V_\alpha = Y - f(X - \smile\{G \mid G \in \mathfrak{G}_\alpha\})$ is an open covering of Y by the closedness of f. Since Y is paracompact, there exists a locally finite open refinement $\{W_\delta \mid \delta \in \Delta\}$ of $\{V_\alpha \mid \alpha \in A\}$. For every W_δ, we can find $V_{\alpha(\delta)}$ of $\{V_\alpha \mid \alpha \in A\}$ such that $W_\delta \subset V_{\alpha(\delta)}$. Then $\{f^{-1}(W_\delta) \frown G \mid G \in \mathfrak{G}_{\alpha(\delta)}; \delta \in \Delta\}$ is locally finite open refinement of $\{U_\lambda\}$. Hence X is paracompact. This completes the proof.

Corollary 2.4. *If f is a closed continuous mapping of a Hausdorff space X onto a paracompact Hausdorff space Y such that the inverse image $f^{-1}(y)$ is paracompact and the boundary $\mathfrak{B} f^{-1}(y)$ is compact for every point y of Y, then X is paracompact and normal.*

Proof. From Corollary 2.2, we can see that the condition (a) is satisfied. The condition (c) follows from the compactness of $\mathfrak{B} f^{-1}(y)$. Hence by Theorem 6, we get Corollary 2.4.

Corollary 2.5. *If f is a closed continuous mapping of a regular topological space X onto a paracompact Hausdorff space Y such that the inverse image $f^{-1}(y)$ is paracompact and normal and the boundary $\mathfrak{B} f^{-1}(y)$ has the Lindelöf property for every point y of Y, then X is paracompact and normal.*

Proof. By Corollary 2.3, we can easily see that X is normal. We next prove that the condition (c) is satisfied. Since $\mathfrak{B} f^{-1}(y)$ has the Lindelöf property, we may consider, as a locally finite open covering, a locally finite countable open covering $\{U_i\}$ of $\mathfrak{B} f^{-1}(y)$. On the other hand, from the proof of Lemma 1 of C. H. Dowker [2], we can see that there exists a locally finite countable open covering $\{V_i\}$ of X such that $V_i \frown A \subset U_i$ for each i. Therefore the condition (c) of Theorem 6 is satisfied. Hence, by Theorem 6, X is paracompact. This completes the proof.

References

[1] H. Tamano: A theorem on closed mappings, Memoirs of the College of Science, Univ. Kyoto, ser. A 33, Math. no. 2, 309–315 (1960).

[2] C. H. Dowker: On a theorem of Hanner, Ark. Math., **2**, 307–317 (1952).