71. Inverse Images of Closed Mappings. I

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Let f be a closed continuous mapping of a topological space X onto a topological space Y. It is well known that if Y is paracompact and $f^{-1}(y)$ is compact for each point y of Y, then X is paracompact [2, 3]. It is interesting to know under what conditions the topological properties of Y may be preserved by the inverse mapping f^{-1}. H. Tamano [7] has recently obtained the necessary and sufficient condition that the inverse image space $X=f^{-1}(Y)$ be normal where X and Y are completely regular T_1-spaces and Y is paracompact.

In this note, we shall investigate the compactness of the inverse image space $X=f^{-1}(Y)$ under the closed continuous mapping f.

In the first place, let us quickly recall some definitions which were introduced by K. Morita [5]. For any infinite cardinal number $\mathfrak{m}$, a topological space X is said to be $\mathfrak{m}$-paracompact if any open covering of X with power $\leqq \mathfrak{m}$ (i.e. consisting of at most $\mathfrak{m}$ sets) admits a locally finite open covering as its refinement. A topological space X is called $\mathfrak{m}$-compact if every open covering of power $\leqq \mathfrak{m}$ has a finite subcovering.

Theorem 1. *If f is a closed continuous mapping of a topological space X onto an $\mathfrak{m}$-paracompact ($\mathfrak{m}$-compact) topological space Y such that the inverse image $f^{-1}(y)$ is $\mathfrak{m}$-compact for every point y of Y, then X is $\mathfrak{m}$-paracompact ($\mathfrak{m}$-compact).*

Proof. Let $\mathfrak{U}=\{U_\lambda \mid \lambda \in \Lambda\}$, $|\Lambda| \leqq \mathfrak{m}$ be an open covering of X where $|\Lambda|$ denotes the power of Λ. And let Γ be the family of all finite subsets γ of Λ, then $|\Gamma| \leqq \mathfrak{m}$. Since $f^{-1}(y)$ is $\mathfrak{m}$-compact for every point y of Y, there exists a finite subset γ of Λ such that $f^{-1}(y) \subset \bigcup_{\lambda \in \gamma} U_\lambda$. Let $V_\gamma = Y - f(X - \bigcup_{\lambda \in \gamma} U_\lambda)$, then V_γ is open by the closedness of f and $y \in V_\gamma$ and $f^{-1}(V_\gamma) \subset \bigcup_{\lambda \in \gamma} U_\lambda$. Therefore $\mathfrak{V}=\{V_\gamma \mid \gamma \in \Gamma\}$ is an open covering of Y with power $\leqq \mathfrak{m}$. If Y is $\mathfrak{m}$-paracompact ($\mathfrak{m}$-compact), then there exists a locally finite (finite) open refinement $\{W_\delta \mid \delta \in \Delta\}$ of $\mathfrak{V}$. Since, for each δ there exists a $\gamma_\delta \in \Gamma$ such that $f^{-1}(W_\delta) \subset f^{-1}(V_{\gamma_\delta}) \subset \bigcup_{\lambda \in \gamma_\delta} U_\lambda$, $\{f^{-1}(W_\delta) \cap U_\gamma \mid \delta \in \Delta, \lambda \in \gamma_\delta\}$ is a locally finite (finite) open refinement of $\mathfrak{U}$. Thus we get the theorem. From Theorem 1, we have the following corollaries (see [2, 3, 1]).

Corollary 1.1. *If f is a closed continuous mapping of a topological space X onto a paracompact topological space Y such that* *the inverse image $f^{-1}(y)$ is compact for every point y of Y, then X is paracompact.*

Corollary 1.2. *If f is a closed continuous mapping of a topological space X onto a countably paracompact topological space Y such that the inverse image $f^{-1}(y)$ is countably compact for every point y of Y, then X is countably paracompact.*

Corollary 1.3 (J. Dieudonné). *If X is a paracompact topological space and Y is a compact topological space, then the product space $X\times Y$ is paracompact.*

Corollary 1.3 follows from the fact that the projection of $X\times Y$ onto X is a closed continuous mapping.

Theorem 2. *Let X be a normal topological space and let Y be a countably paracompact collectionwise normal topological space. If $f(X)=Y$ is a closed continuous mapping such that the inverse image $f^{-1}(y)$ is compact for every point y of Y, then X is countably paracompact and collectionwise normal.*

Proof. To prove this theorem, we shall use the theorem due to M. Katětov [4, Theorem 4.1]. Let $\{X_\alpha\}$ be a locally finite collection of subsets of X, then we can see that $\{f(X_\alpha)\}$ is locally finite in Y. In fact, let y be any point of Y. Since $f^{-1}(y)$ is compact and $\{X_\alpha\}$ is locally finite, there exists an open set G such that $f^{-1}(y)\subset G$ and G intersects only a finite number of X_α. Then $U(y)=Y-f(X-G)$ is an open neighborhood of y which intersects only a finite number of $f(X_\alpha)$. Hence $\{f(X_\alpha)\}$ is locally finite in Y. Since Y is countably paracompact and collectionwise normal, there exists a locally finite collection of open sets $\{G_\alpha\}$ such that $G_\alpha\supset f(X_\alpha)$ for every α. Hence we get $f^{-1}(G_\alpha)\supset X_\alpha$ and $\{f^{-1}(G_\alpha)\}$ is a locally finite collection of open sets. Therefore X is countably paracompact and collectionwise normal. This completes the proof.

Theorem 3. *If X is a topological space such that every point of X has a neighborhood basis with power $\leqq \mathfrak{m}$ and Y is an $\mathfrak{m}$-compact topological space, then the projection of the product space $X\times Y$ onto X is closed.*

Proof. Let $f(x,y)=x$ be the projection of $X\times Y$ onto X and let A be a closed subset of $X\times Y$. Let x_0 be any point of $\overline{f(A)}$ and let $\{U_\lambda(x_0)\mid \lambda\in\Lambda\}, |\Lambda|\leqq\mathfrak{m}$ be a neighborhood basis of x_0, then $U_\lambda(x_0)\frown f(A)\neq\phi$. Hence $f^{-1}(U_\lambda(x_0))\frown A\neq\phi$ for every $\lambda\in\Lambda$. On the other hand, since $f^{-1}(U_\lambda(x_0))=U_\lambda(x_0)\times Y$, we have $(U_\lambda(x_0)\times Y)\frown A\neq\phi$, Hence $A[U_\lambda(x_0)]\neq\phi$ where $A[U_\lambda(x_0)]$ denotes the set $\{y\mid (x',y)\in A$ for some $x'\in U_\lambda(x_0)\}$. Since $\{\overline{A[U_\lambda(x_0)]}\mid\lambda\in\Lambda\}$ has the finite intersection property in Y and Y is $\mathfrak{m}$-compact, we have $\underset{\lambda}{\frown}\overline{A[U_\lambda(x_0)]}\neq\phi$. On the other hand, we can easily see that $\underset{\lambda}{\frown}\overline{A[U_\lambda(x_0)]}=A[x_0]$. Hence $A[x_0]\neq\phi$.

Let y_0 be a point of $A[x_0]$, then $(x_0, y_0) \in A$. Hence $x_0 \in f(A)$ that is, $f(A)$ is closed. This completes the proof.

By Theorems 1 and 3, we can easily verify the following corollaries.

Corollary 1.4. *If X is an $\mathfrak{m}$-paracompact ($\mathfrak{m}$-compact) topological space such that every point of X has a neighborhood basis with power $\leqq \mathfrak{m}$ and Y is an $\mathfrak{m}$-compact topological space, then the product space $X \times Y$ is $\mathfrak{m}$-paracompact ($\mathfrak{m}$-compact).*

Corollary 1.5. *If X is a countably compact topological space which satisfies the first countability axiom and Y is a countably compact topological space, then the product space $X \times Y$ is countably compact.*

Remark 1.1. It was shown by examples that the product space of two countably compact topological spaces is not always countably compact [6, 8]. The above corollary has some interest as the case of the product space is countably compact.

Corollary 1.6. *If X is a paracompact topological space such that every point of X has a neighborhood basis with power $\leqq \mathfrak{m}$ and Y is an $\mathfrak{m}$-compact topological space, then the product space $X \times Y$ is $\mathfrak{m}$-paracompact.*

Theorem 4. *Let X be a topological space such that there exists an F_σ-set which is not closed and let Y be a topological space. If the projection of the product space $X \times Y$ onto X is closed, then Y is countably compact.*

Proof. Let $\bigcup_{i=1}^{\infty} A_i$ be not closed where each A_i is closed. Let Y be not countably compact. Then there exists a decreasing sequence $\{B_i\}$ of closed subsets of Y such that $\bigcap_{i=1}^{\infty} B_i = \phi$. Let $C = \bigcup_{i=1}^{\infty} (A_i \times B_i)$, then we can see that C is a closed subset of $X \times Y$. In fact, since $\{B_i\}$ is locally finite in Y, $\{A_i \times B_i \mid i=1, 2, \cdots\}$ is locally finite in $X \times Y$. Hence C is a closed subset of $X \times Y$. Now let p be the projection of $X \times Y$ onto X, then $p(x, y) = x$ for every point (x, y) of $X \times Y$. Then $p(C) = \bigcup_{i=1}^{\infty} A_i$ is not closed. Therefore the projection p is not closed. This completes the proof.

Corollary 1.7. *Let X be a non-discrete T_1-space which satisfies the first countability axiom and let Y be a topological space. Then the projection p of the product space $X \times Y$ onto X is closed if and only if Y is countably compact.*

Proof. The "if" part is an immediate consequence of Theorem 3. We shall show the "only if" part. Since X is a non-discrete T_1-space which satisfies the first countability axiom, there exist a point x_0 and a sequence $\{x_n\}$ of X such that $x_n \to x_0, x_n \neq x_0$ for $n = 1, 2 \cdots$.

Then the set $\bigcup_{n=1}^{\infty} x_n$ is not closed and each x_n is closed since X is a T_1-space. Then, by Theorem 4, Y is countably compact. This completes the proof.

Remark 1.2. H. Tamano [7] stated that if the projection of the product space $X\times Y$ onto X is closed, then Y is pseudocompact provided that X and Y are completely regular T_1-spaces and X is not a finite set. But his proof has a slight careless error. By use of Theorem 4, we can establish the following theorem. *Let X be a topological space such that there exists a real valued continuous function $f(x)$ on X and the set $\{x \mid f(x)>0\}$ is not closed and let Y be a topological space. If the projection of the product space $X\times Y$ onto X is closed, then Y is countably compact.*

In fact, let $A_i=\left\{x \,\middle|\, \frac{1}{i} \geqq f(x) \geqq \frac{1}{i+1}\right\}$, then $\bigcup_{i=1}^{\infty} A_i$ is not closed and each A_i is closed. Hence, by Theorem 4, Y is countably compact. So that Theorem 4 is better than his attempt, since the pseudocompactness follows from the countable compactness.

References

[1] J. Dieudonné: Une généralization des espaces compacts, J. Math. Pures Appl., **23**, 65-76 (1944).

[2] S. Hanai: On closed mappings. II, Proc. Japan Acad., **32**, 388-391 (1956).

[3] M. Henriksen and J. R. Isbell: Some properties of compactifications, Duke Math. J., **25**, 83-105 (1958).

[4] M. Katětov: Extension of locally finite coverings, Colloq. Math., **6**, 141-151 (1958).

[5] K. Morita: Paracompactness and product spaces, forthcoming.

[6] J. Novák: On the cartesian product spaces, Fund. Math., **40**, 106-112 (1953).

[7] H. Tamano: A theorem on closed mappings, Memoirs of the College of Science, Univ. Kyoto, ser. A 33, Math. no. 2, 309-315 (1960).

[8] H. Terasaka: On cartesian product of compact spaces, Osaka Math. J., **4**, 11-15 (1952).