68. On Köthe's Problem concerning Algebras for which Every Indecomposable Module Is Cyclic. I[1]

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§1. **Introduction.** In 1934, in connection with his theory, G. Köthe [2] proposed the problem to determine the general type of ring A (with a unit and satisfying the minimum condition) possessing the property that every finitely generated indecomposable left or right A-module is cyclic. A ring or algebra with this property will be called a Köthe ring or Köthe algebra. Köthe himself solved this problem for the special case of commutative rings. As for non-commutative rings, he proved only that uni-serial rings are Köthe rings.

In 1941, T. Nakayama [5, 6] introduced the notion of generalized uni-serial rings as a generalization of uni-serial rings, and proved that generalized uni-serial rings are Köthe rings. However, as is shown by Nakayama, the rings of this type are not general enough for solving Köthe's problem.

Recently H. Tachikawa [8] has called an algebra A an algebra of cyclic-cocyclic representation type, if any finitely generated indecomposable left (resp. right) A-module is either homomorphic to an indecomposable left (resp. right) ideal of A generated by a primitive idempotent, or isomorphic to a submodule of an indecomposable injective left (resp. right) module, and he has determined the structure of such algebras. However, algebras of cyclic-cocyclic representation type are not always Köthe algebras.

Thus the most general type of Köthe rings known hitherto is generalized uni-serial rings, and any class of rings which contains non-commutative rings and for which the solution of Köthe's problem is given seems to have never been obtained in the literature.[2]

The purpose of this paper is to announce that Köthe's problem mentioned above is completely solved for the case of self-basic algebras.[3] As is well known, every commutative algebra is self-basic.

1) The results of this paper were reported by the author at the meeting of Math. Soc. of Japan, held in October, 1960.

2) In case A is an algebra over an algebraically closed field and the square of its radical is zero, T. Yoshii [10] has given some sufficient conditions for A to be a Köthe algebra.

3) An algebra (resp. ring) A is called a self-basic algebra (resp. ring) if A is the basic ring of A itself.

Accordingly, our solution is, as far as algebras are concerned, the first result in the study of Köthe's problem for the case of non-commutative rings. Our precise results are stated in §§2 and 3 where a more general case is treated and the phrase "as far as algebras are concerned" is shown to be not necessary in the above sentence.

The author wishes to express his hearty thanks to Professor K. Morita for his constant encouragement and criticism during this work.

§ 2. **Statement of the results, I.** Throughout this paper A will be assumed to be a ring which has a unit 1 and satisfies the minimum condition for left ideals, and we shall denote by N the radical of A. By a left (resp. right) A-module and a homomorphism we shall mean respectively a unital left (resp. right) A-module and an A-homomorphism. As is defined in the introduction, a ring A is called a Köthe ring if it satisfies next two conditions:

(a) Every finitely generated indecomposable left A-module is homomorphic to A.

(a*) Every finitely generated indecomposable right A-module is homomorphic to A.

In order to state our results we shall define here some notions related to A-modules and A-homomorphisms.[4] Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be left A-modules and $\mathfrak{M}_1, \mathfrak{N}_1, \mathfrak{M}_2$ and $\mathfrak{N}_2$ A-submodules such that $\mathfrak{L}_1 \supset \mathfrak{M}_1 \supsetneqq \mathfrak{N}_1$ and $\mathfrak{L}_2 \supset \mathfrak{M}_2 \supset \mathfrak{N}_2$. If for a given homomorphism $\varphi: \mathfrak{N}_1 \to \mathfrak{N}_2$[5] there exists a homomorphism $\Phi: \mathfrak{M}_1 \to \mathfrak{M}_2$ which coincides with φ on $\mathfrak{N}_1$, then φ is said to be extendable to Φ, and Φ is called an extension of φ. If φ has no extension such that $S(\mathfrak{M}_1) = S(\mathfrak{N}_1)$,[6] then φ is said to be restrictedly maximal. Furthermore if φ has no extension without any restriction, then φ is merely said to be maximal. Next, a left A-module $\mathfrak{M}(\neq 0)$ will be said to be quasi-primitive if it is homomorphic to a primitive left ideal of A. More strongly, a left A-module $\mathfrak{M}(\neq 0)$ is called a uni-serial module if it has a unique composition series.

For the case of self-basic algebras our characterization of Köthe algebra will be stated as follows:

Let A be a finite dimensional self-basic algebra over a commutative field, N the radical of A and $e_\kappa, e_\lambda, e_\mu, \cdots$ the primitive idempotents of A. Then in order that A be a Köthe algebra it is necessary and sufficient that the following conditions are satisfied:

I. Let $Ae_\lambda g$ be an arbitrary quasi-primitive left A-module.[7]

4) As to denomination of most of these notions we shall follow H. Tachikawa [7].

5) "Homomorphism $\varphi: \mathfrak{N}_1 \to \mathfrak{N}_2$" means a homomorphism φ of $\mathfrak{N}_1$ into $\mathfrak{N}_2$.

6) For a left A-module $\mathfrak{M}$ we always denote its socle by $S(\mathfrak{M})$; $S(\mathfrak{M}) = \{u \mid Nu = 0, u \in \mathfrak{M}\}$.

7) By $Ae_\lambda g$ we always mean a quasi-primitive left A-module with a generator g.

Then its socle is isomorphic either to a simple left A-module or to a direct sum of two simple left A-modules which are not isomorphic to each other; that is, we have either $S(Ae_\lambda g)\approx Ae_\kappa/Ne_\kappa$ or $S(Ae_\lambda g)\approx Ae_\kappa/Ne_\kappa\oplus Ae_\rho/Ne_\rho$, with $Ae_\kappa\not\approx Ae_\rho$.

II. Assume that both $Ae_\lambda g_1$ and $Ae_\lambda g_2$ are not simple and $S(Ae_\lambda g_1)\approx S(Ae_\lambda g_2)\approx Ae_\kappa/Ne_\kappa$. Let φ be an isomorphism of $S(Ae_\lambda g_1)$ onto $S(Ae_\lambda g_2)$. Then either φ is extendable to a monomorphism Φ_1: $Ae_\lambda g_1\rightarrow Ae_\lambda g_2$, or φ^{-1} is extendable to a monomorphism Φ_2: $Ae_\lambda g_2\rightarrow Ae_\lambda g_1$.

III. Assume that $S(Ae_{\lambda_1}g_1)\approx S(Ae_{\lambda_2}g_2)\approx S(Ae_{\lambda_3}g_3)\approx Ae_\kappa/Ne_\kappa$. Then there exists at least one monomorphism which maps one of $Ae_{\lambda_1}g_1$, $Ae_{\lambda_2}g_2$ and $Ae_{\lambda_3}g_3$ into one of the others.

IV. Assume that $S(Ae_{\lambda_1}g_1)\approx S(Ae_{\lambda_2}g_2)\approx Ae_\sigma/Ne_\sigma\oplus Ae_\tau/Ne_\tau$, i.e. $S(Ae_{\lambda_1}g_1)=Ae_\sigma u_1e_{\lambda_1}g_1\oplus Ae_\tau v_1e_{\lambda_1}g_1(u_1, v_1$ in $N)$ and $S(Ae_{\lambda_2}g_2)=Ae_\sigma u_2e_{\lambda_2}g_2\oplus Ae_\tau v_2e_{\lambda_2}g_2(u_2, v_2$ in $N)$. Let φ be an isomorphism of $Ae_\sigma u_1e_{\lambda_1}g_1$ onto $Ae_\sigma u_2e_{\lambda_2}g_2$. Then either φ is extendable to a homomorphism Φ_1: $Ae_{\lambda_1}g_1\rightarrow Ae_{\lambda_2}g_2$, or φ^{-1} is extendable to a homomorphism Φ_2: $Ae_{\lambda_2}g_2\rightarrow Ae_{\lambda_1}g_1$.

V. Assume that $S(Ae_{\lambda_1}g_1)\approx S(Ae_{\lambda_2}g_2)\approx Ae_\kappa/Ne_\kappa$, $e_\kappa Ne_\kappa Ae_{\lambda_1}g_1\neq 0$ and $e_\kappa Ne_\kappa Ae_{\lambda_2}g_2\neq 0$. Let φ be an isomorphism of $S(Ae_{\lambda_1}g_1)$ onto $S(Ae_{\lambda_2}g_2)$. Then either φ is extendable to a monomorphism Φ_1: $Ae_{\lambda_1}g_1\rightarrow Ae_{\lambda_2}g_2$, or φ^{-1} is extendable to a monomorphism Φ_2: $Ae_{\lambda_2}g_2\rightarrow Ae_{\lambda_1}g_1$.

VI. Assume that $e_\lambda Ne_\lambda g\neq 0$. Then $Ae_\lambda Ne_\lambda g$ is a uni-serial module.

VII. Assume that $S(Ae_{\lambda_1}g_1)\approx S(Ae_{\lambda_2}g_2)\approx Ae_\kappa/Ne_\kappa$ and assume that the isomorphism φ: $S(Ae_{\lambda_1}g_1)\rightarrow S(Ae_{\lambda_2}g_2)$ is extendable to a monomorphism Φ: $\mathfrak{L}_1\rightarrow\mathfrak{L}_2$ where $\mathfrak{L}_1\subset Ne_{\lambda_1}g_1$ and $\mathfrak{L}_2\subset Ne_{\lambda_2}g_2$. If Φ is maximal, then it is impossible that $Ae_{\lambda_1}Ne_{\lambda_1}g_1$ properly includes $\mathfrak{L}_1$.

VIII. Assume that $Ae_{\lambda_1}g_1$ is a module such that $Ae_{\lambda_1}g_1/N^le_{\lambda_1}g_1$ $(l\geqq 1)$ is uni-serial, $N^le_{\lambda_1}g_1=Ae_\kappa te_{\lambda_1}g_1\oplus Ae_\rho we_{\lambda_1}g_1(t, w$ in $N^l)$ where $Ae_\kappa te_{\lambda_1}g_1$ is uni-serial, and $S(Ae_{\lambda_1}g_1)=N^me_\kappa te_{\lambda_1}g_1\oplus Ae_\rho we_{\lambda_1}g_1(m\geqq 1)$. Let $Ae_{\lambda_2}g_2$ be another module whose socle is isomorphic to $N^me_\kappa te_{\lambda_1}g_1$. If the isomorphism φ: $S(Ae_{\lambda_2}g_2)\rightarrow S(Ae_{\lambda_1}g_1/Ae_\rho we_{\lambda_1}g_1)$ is extendable, then either φ is extendable to a monomorphism Φ_1: $Ae_{\lambda_2}g_2\rightarrow Ae_{\lambda_1}g_1/Ae_\rho we_{\lambda_1}g_1$, or φ^{-1} is extendable to a monomorphism Φ_2: $Ae_{\lambda_1}g_1/Ae_\rho we_{\lambda_1}g_1\rightarrow Ae_{\lambda_2}g_2$.

IX. Assume that $Ae_{\lambda_1}g_1$ is a module such that $Ae_{\lambda_1}g_1/N^le_{\lambda_1}g_1$ $(l\geqq 2)$ is uni-serial, $N^le_{\lambda_1}g_1=Ae_\kappa te_{\lambda_1}g_1\oplus Ae_\rho we_{\lambda_1}g_1(t, w$ in $N^l)$ where $Ae_\kappa te_{\lambda_1}g_1$ is uni-serial, and $S(Ae_{\lambda_1}g_1)=N^me_\kappa te_{\lambda_1}g_1\oplus Ae_\rho we_{\lambda_1}g_1(m\geqq 0)$.[8] Assume that $Ae_{\lambda_2}g_2$ is another non-simple module whose socle is isomorphic to $N^me_\kappa te_{\lambda_1}g_1$. Let φ be an isomorphism of $S(Ae_{\lambda_2}g_2)$ onto $S(Ae_{\lambda_1}g_1/Ae_\rho we_{\lambda_1}g_1)$. Then either φ is extendable to a monomorphism Φ_1: $Ae_{\lambda_2}g_2\rightarrow Ae_{\lambda_1}g_1/Ae_\rho we_{\lambda_1}g_1$, or φ^{-1} is extendable to a monomorphism Φ_2: $Ae_{\lambda_1}g_1/Ae_\rho we_{\lambda_1}g_1\rightarrow Ae_{\lambda_2}g_2$, or there exists a maximal extension of φ^{-1} which maps $Ne_{\lambda_1}g_1/Ae_\rho we_{\lambda_1}g_1$ into $Ne_{\lambda_2}g_2$.

8) By N^0 we shall always mean A itself.

X. Assume that $Ae_{\lambda_1}g_1$ is a module such that $Ne_{\lambda_1}g_1=Ae_\varepsilon te_{\lambda_1}g_1+Ae_\rho we_{\lambda_1}g_1(t, w \text{ in } N)$ where $Ae_\varepsilon te_{\lambda_1}g_1$ is uni-serial, $Ae_\varepsilon te_{\lambda_1}g_1\frown Ae_\rho we_{\lambda_1}g_1=N^m e_\varepsilon te_{\lambda_1}g_1=Ae_\sigma ue_\rho we_{\lambda_1}g_1\neq 0$ ($m\geqq 1$ and u in N), $Ne_\rho we_{\lambda_1}g_1=Ae_\sigma ue_\rho we_{\lambda_1}g_1\oplus Ae_\tau ve_\rho we_{\lambda_1}g_1(v \text{ in } N)$ where $Ae_\tau vwe_{\lambda_1}g_1$ is uni-serial, and $S(Ae_{\lambda_1}g_1)=Ae_\sigma uwe_{\lambda_1}g_1\oplus N^k e_\tau vwe_{\lambda_1}g_1(k\geqq 0)$. Assume that $Ae_{\lambda_2}g_2$ is a non-simple module whose socle is isomorphic to $N^k e_\tau vwe_{\lambda_1}g_1$. Let φ be an isomorphism which maps $S(Ae_{\lambda_2}g_2)$ onto $N^k e_\tau vwe_{\lambda_1}g_1+Ae_\varepsilon te_{\lambda_1}g_1/Ae_\varepsilon te_{\lambda_1}g_1$ considered as a submodule of $Ae_{\lambda_1}g_1/Ae_\varepsilon te_{\lambda_1}g_1$. Then φ is extendable; more precisely, either φ is extendable to a monomorphism Φ_1: $Ae_{\lambda_2}g_2\rightarrow Ae_{\lambda_1}g_1/Ae_\varepsilon te_{\lambda_1}g_1$, or φ^{-1} is extendable to a monomorphism Φ_2: $Ae_{\lambda_1}g_1/Ae_\varepsilon te_{\lambda_1}g_1\rightarrow Ae_{\lambda_2}g_2$.

XI. Assume that $Ae_\lambda g/N^3e_\lambda g$ is uni-serial and $N^3e_\lambda g=Ae_\varepsilon te_\lambda g\oplus Ae_\rho we_\lambda g(t, w \text{ in } N^3)$ where $e_\varepsilon te_\lambda g\neq 0$ and $e_\rho we_\lambda g\neq 0$. Then we have both $Ne_\varepsilon te_\lambda g=0$ and $Ne_\rho we_\lambda g=0$, i.e. $N^4e_\lambda g=0$.

XII. Assume that $Ae_\lambda g/N^2e_\lambda g$ is uni-serial and $N^2e_\lambda g=Ae_\varepsilon te_\lambda g\oplus Ae_\rho we_\lambda g(t, w \text{ in } N^2)$ where $e_\varepsilon te_\lambda g\neq 0$ and $e_\rho we_\lambda g\neq 0$. Then we have either $Ne_\varepsilon te_\lambda g=0$ or $Ne_\rho we_\lambda g=0$.

XIII. Assume that $Ae_{\lambda_1}g_1$ is a uni-serial module and $S(Ae_{\lambda_1}g_1)=N^4e_{\lambda_1}g_1$. Then it is impossible that there exists a uni-serial module $Ae_{\lambda_2}g_2$ such that $Ae_{\lambda_2}g_2\not\approx Ne_{\lambda_1}g_1$ but $Ne_{\lambda_2}g_2\approx N^2e_{\lambda_1}g_1$.

XIV. Assume that $Ae_{\lambda_1}g_1$ is a uni-serial module and $S(Ae_{\lambda_1}g_1)=N^3e_{\lambda_1}g_1$. Then it is impossible that there exists a uni-serial module $Ae_{\lambda_2}g_2$ such that $Ne_{\lambda_2}g_2\not\approx Ne_{\lambda_1}g_1$ but $N^2e_{\lambda_2}g_2\approx N^2e_{\lambda_1}g_1$.

XV. Assume that $Ae_{\lambda_i}g_i$, $i=1, 2,\cdots, n$ ($n\geqq 3$), are modules each of which satisfies the following conditions:

(i) $Ae_{\lambda_i}\not\approx Ae_{\lambda_j}$ if $i\neq j$.

(ii) $Ne_{\lambda_i}g_i=Ae_{\varepsilon_i}t_ie_{\lambda_i}g_i\oplus Ae_{\rho_i}w_ie_{\lambda_i}g_i(t_i, w_i \text{ in } N)$ where $Ae_{\varepsilon_i}t_ie_{\lambda_i}g_i$ as well as $Ae_{\rho_i}w_ie_{\lambda_i}g_i$ is uni-serial.

(iii) $S(Ae_{\lambda_i}g_i)=Ae_{\sigma_i}u_ie_{\varepsilon_i}t_ie_{\lambda_i}g_i\oplus Ae_{\sigma_{i+1}}v_ie_{\rho_i}w_ie_{\lambda_i}g_i(u_i, v_i \text{ in } A)$ where $Ae_{\sigma_i}\not\approx Ae_{\sigma_j}$ if $i<j\leqq n$ but $Ae_{\sigma_1}\approx Ae_{\sigma_{n+1}}$.

(iv) Homomorphisms φ_i: $Ae_{\sigma_{i+1}}v_iw_ie_{\lambda_i}g_i\rightarrow Ae_{\sigma_{i+1}}u_{i+1}t_{i+1}e_{\lambda_{i+1}}g_{i+1}$, $i=1, 2,\cdots, n-1$, are all restrictedly maximal and φ_i^{-1}, $i=1, 2,\cdots, n-1$, are also restrictedly maximal.

Then there exists at least one i such that either φ_i is extendable to a homomorphism Φ_1: $Ae_{\lambda_i}g_i\rightarrow Ae_{\lambda_{i+1}}g_{i+1}$, or φ_i^{-1} is extendable to a homomorphism Φ_2: $Ae_{\lambda_{i+1}}g_{i+1}\rightarrow Ae_{\lambda_i}g_i$.

XVI. Assume that $Ae_{\lambda_i}g_i$, $i=1, 2,\cdots, n$ ($n\geqq 3$), are modules each of which satisfies the following conditions:

(i) $Ae_{\lambda_i}\not\approx Ae_{\lambda_j}$ if $i<j\leqq n-1$, but $Ae_{\lambda_1}\approx Ae_{\lambda_n}$.

(ii) $Ne_{\lambda_i}g_i=Ae_{\varepsilon_i}t_ie_{\lambda_i}g_i\oplus Ae_{\rho_i}w_ie_{\lambda_i}g_i(t_i, w_i \text{ in } N)$, $2\leqq i\leqq n-1$, where $Ae_{\varepsilon_i}t_ie_{\lambda_i}g_i$ as well as $Ae_{\rho_i}w_ie_{\lambda_i}g_i$ is uni-serial, but both $Ae_{\lambda_1}g_1$ and $Ae_{\lambda_n}g_n$ are non-simple uni-serial modules where $Ne_{\lambda_1}g_1=Ae_{\rho_1}w_1e_{\lambda_1}g_1(w_1 \text{ in } N)$ and $Ne_{\lambda_n}g_n=Ae_{\varepsilon_n}t_ne_{\lambda_n}g_n(t_n \text{ in } N)$.

(iii) $S(Ae_{\lambda_i}g_i)=Ae_{\sigma_i}u_ie_{\kappa_i}t_ie_{\lambda_i}g_i\oplus Ae_{\sigma_{i+1}}v_ie_{\rho_i}w_ie_{\lambda_i}g_i(u_i,\ v_i$ in $A)$, $2\leqq i\leqq n-1$, where $Ae_{\sigma_i}\not\approx Ae_{\sigma_j}$ if $i\neq j$, but $S(Ae_{\lambda_1}g_1)=Ae_{\sigma_2}v_1e_{\rho_1}w_1e_{\lambda_1}g_1(v_1$ in $A)$ and $S(Ae_{\lambda_n}g_n)=Ae_{\sigma_n}u_ne_{\kappa_n}t_ne_{\lambda_n}g_n(u_n$ in $A)$.

(iv) Homomorphisms φ_i: $Ae_{\sigma_{i+1}}v_iw_ie_{\lambda_i}g_i\to Ae_{\sigma_{i+1}}u_{i+1}t_{i+1}e_{\lambda_{i+1}}g_{i+1}$, $i=1, 2,\cdots, n-1$, are all restrictedly maximal and φ_i^{-1}, $i=1, 2,\cdots, n-1$, are also restrictedly maximal.

Then there exists at least one i such that either φ_i is extendable to a homomorphism Φ_1: $Ae_{\lambda_i}g_i\to Ae_{\lambda_{i+1}}g_{i+1}$, or φ_i^{-1} is extendable to a homomorphism Φ_2: $Ae_{\lambda_{i+1}}g_{i+1}\to Ae_{\lambda_i}g_i$.

XVII. Assume that $Ae_{\lambda_1}g_1$ is a module such that $Ne_{\lambda_1}g_1=Ae_\kappa te_{\lambda_1}g_1\oplus Ae_\rho we_{\lambda_1}g_1(t, w$ in $N)$ where $Ae_\kappa te_{\lambda_1}g_1$ is uni-serial, $Ae_\kappa te_{\lambda_1}g_1\frown Ae_\rho we_{\lambda_1}g_1=Ne_\kappa te_{\lambda_1}g_1=Ae_\sigma ue_\rho we_{\lambda_1}g_1\neq 0$ $(u$ in $N)$, and $S(Ae_{\lambda_1}g_1)=Ne_\rho we_{\lambda_1}g_1=Ae_\sigma ue_\rho we_{\lambda_1}g_1\oplus Ae_\tau ve_\rho we_{\lambda_1}g_1(v$ in $N)$ where $e_\tau vwe_{\lambda_1}g_1\neq 0$. Assume that $Ae_{\lambda_2}g_2$ is a uni-serial module whose socle is isomorphic to Ae_σ/Ne_σ. Let φ be an isomorphism which maps $S(Ae_{\lambda_2}g_2)$ onto $Ae_\sigma uwe_{\lambda_1}g_1+Ae_\tau vwe_{\lambda_1}g_1/Ae_\tau vwe_{\lambda_1}g_1$ considered as a submodule of $Ae_{\lambda_1}g_1/Ae_\tau vwe_{\lambda_1}g_1$.

Then there exists no maximal extension of φ^{-1} which maps $Ae_\kappa te_{\lambda_1}g_1+Ae_\tau vwe_{\lambda_1}g_1/Ae_\tau vwe_{\lambda_1}g_1$ into $Ne_{\lambda_2}g_2$.

XVIII. Assume that $Ae_\lambda g$ is a module such that $Ae_\lambda g/N^2e_\lambda g$ is uni-serial, $N^2e_\lambda g=Ae_\kappa te_\lambda g+Ae_\rho we_\lambda g(t, w$ in $N^2)$, $Ae_\kappa te_\lambda g\frown Ae_\rho we_\lambda g=Ne_\kappa te_\lambda g=Ae_\sigma ue_\rho we_\lambda g\neq 0$ $(u$ in $N)$, and $Ne_\rho we_\lambda g=Ae_\sigma ue_\rho we_\lambda g\oplus Ae_\tau ve_\rho we_\lambda g(v$ in $N)$ where $Ae_\tau vwe_\lambda g$ is uni-serial and $Ne_\sigma uwe_\lambda g=0$. Then $Ne_\tau vwe_\lambda g=0$ holds.

XIX. Assume that $Ae_{\lambda_1}g_1$ is a module such that $Ne_{\lambda_1}g_1=Ae_\kappa te_{\lambda_1}g_1+Ae_\rho we_{\lambda_1}g_1(t, w$ in $N)$, $Ae_\kappa te_{\lambda_1}g_1\frown Ae_\rho we_{\lambda_1}g_1=Ne_\kappa te_{\lambda_1}g_1=Ne_\rho we_{\lambda_1}g_1=Ae_\sigma ue_\kappa te_{\lambda_1}g_1$ $(u$ in $N)$, and $Ne_\sigma ute_{\lambda_1}g_1$ is simple. Assume that $Ae_{\lambda_2}g_2$ is a uni-serial module whose socle is isomorphic to $Ne_\sigma ute_{\lambda_1}g_1(=S(Ae_{\lambda_1}g_1))$. Let φ be an isomorphism of $S(Ae_{\lambda_1}g_1)$ onto $S(Ae_{\lambda_2}g_2)$. Then there exists no maximal extension of φ which maps $Ae_\kappa te_{\lambda_1}g_1$ into $N^2e_{\lambda_2}g_2$.

§3. Statement of the results, II. Throughout this paper, for a ring A we shall denote by A° the basic ring of A, by $e_\lambda, e_\mu, e_\nu,\cdots$ the primitive idempotents of A, and by e the unit of A° respectively.

In case there exists a duality between the category of all finitely generated left A-modules and that of all finitely generated right A-modules (this is certainly the case if A is an algebra or a commutative ring (cf. K. Morita [4])), the condition (a*) in §2 is equivalent to the condition (b) below:

(b) The socle of every finitely generated indecomposable left A-module is homomorphic to A.

Now we consider the following conditions (a°) and (b°) in place of (a) and (b):

(a°) Every finitely generated indecomposable left A-module is homomorphic to Ae.

(b°) The socle of every finitely generated indecomposable left A-module is homomorphic to Ae.

In case A is a self-basic ring, (a°) and (b°) coincide with (a) and (b) respectively. More generally, these properties are preserved by the category-isomorphism between categories of all finitely generated left A-modules and of all finitely generated left A°-modules, introduced by K. Morita [4]. Thus we are led to a problem: How we can characterize those rings which satisfy (a°) and (b°)? Our solution to this problem is as follows:

Theorem. *Let A be an associative ring which possesses a unit and satisfies the minimum condition for left ideals. In order that A satisfy* (a°) *and* (b°), *it is necessary and sufficient that A satisfies all the conditions* I-XIX *stated in* §2.

The result of §2 is an immediate consequence of this theorem, and Köthe's solution for commutative case is included in this theorem.

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