65. On Neumann Problem for Laplace-Beltrami Operators

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§1. **Introduction.** In this paper, we deal with the second boundary value problem (Neumann problem) in compact subdomains of Riemannian spaces. Let $\boldsymbol{M}$ be an m-dimensional orientable Riemannian manifold of class C^3 with metric tensor $\|g_{ij}(x)\|$, and let D be a subdomain of $\boldsymbol{M}$ whose closure $\overline{D}$ is compact and whose boundary S consists of a finite number of $(m-1)$-dimensional hypersurfaces of class C^3. We denote by Λ the Laplace-Beltrami operator with respect to $\|g_{ij}(x)\|$:

$$\Lambda u(x)=\frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}\left[\sqrt{g(x)}\,g^{ij}(x)\frac{\partial u(x)}{\partial x^j}\right] \quad \text{for} \quad u\in C^2(D)$$

where $\|g^{ij}(x)\|=\|g_{ij}(x)\|^{-1}$ and $g(x)=\det\|g_{ij}(x)\|$, and by $\frac{\partial}{\partial \boldsymbol{n}}$ the outer normal derivative at any point on the boundary S of D.

Consider the second boundary value problem in D associated with Λ:

$$\Lambda u=f \text{ in } D,\ \frac{\partial u}{\partial \boldsymbol{n}}=\varphi \text{ on } S, \tag{1.1}$$

where f and φ are given functions defined in D and on S respectively. The fundamental solution $U(t, x, y)$ of the initial-boundary value problem of the parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}=\Lambda u+f & (t>0,\ x\in D) \\ u|_{t=0}=u_0,\ \dfrac{\partial u}{\partial \boldsymbol{n}}=\varphi & (\text{on } S) \end{cases} \tag{1.2}$$

is given in [2] (see also [1]). We shall show that the kernel function $K(x, y)$ of the boundary value problem (1.1) is given by

$$K(x,y)=\int_0^\infty\{U(t,x,y)-|D|^{-1}\}\,dt \quad \text{whenever } x\neq y \tag{1.3}$$

where $|D|$ denotes the volume of D. Corresponding results in the case of Dirichlet problem, or in the case where Λ in (1.1) is replaced by $\Lambda-c(x)$ (here $c(x)$ is non-negative and not identically zero), are contained in [2; §10]; in these cases, the term $-|D|^{-1}$ in (1.3) should be omitted.

§2. **Main results.** In order that the boundary value problem (1.1) has a solution, the following condition is necessary:

$$\int_D f(x)\,dx=\int_S \varphi(x)\,dS_x \tag{2.1}$$

where $dx\,(=\sqrt{g(x)}\,dx^1\cdots dx^m)$ and dS_x respectively denote the volume element in D and the hypersurface element on S with respect to the metric given by $\|g_{ij}(x)\|$. The necessity of (2.1) and the uniqueness of solution of (1.1) up to an additive constant may be verified by Green's formula. We shall prove the following theorems.

THEOREM 1. *Let $U(t, x, y)$ be the fundamental solution of the initial-boundary value problem* (1.2). *Then*

$$K(x, y)=\int_0^\infty \{U(t, x, y)-|D|^{-1}\}dt \tag{2.2}$$

exists whenever $x\in\overline{D}$, $y\in\overline{D}$ and $x\neq y$, and satisfies

$$K(x, y)=K(y, x), \tag{2.3}$$

and the following quantities are finite:

$$M_1=\sup_{x\in\overline{D}}\int_D |K(x, y)|\,dy,\quad M_2=\sup_{x\in\overline{D}}\int_S |K(x, y)|\,dS_y. \tag{2.4}$$

THEOREM 2. *Assume that $f(x)$ is Hölder-continuous in D and satisfies $\int_D |f(x)|^2 dx<\infty$, that $\varphi(x)$ is Hölder-continuous on S and that* (2.1) *holds. Then the boundary value problem* (1.1) *has a solution $u(x)$ which is given as follows:*

$$u(x)=-\int_D K(x, y)f(y)\,dy+\int_S K(x, y)\varphi(y)\,dS_y. \tag{2.5}$$

We define the following notations whenever the right-hand side of each formula makes sense:

$$(u, v)=\int_D u(x)\,v(x)\,dx$$

$$\|u\|=\left\{\int_D |u(x)|^2\,dx\right\}^{\frac{1}{2}},\quad \|u\|_S=\left\{\int_S |u(x)|^2\,dS_x\right\}^{\frac{1}{2}},$$

$$\|\nabla u\|=\left\{\int_D g^{ij}(x)\frac{\partial u(x)}{\partial x^i}\frac{\partial u(x)}{\partial x^j}\,dx\right\}^{\frac{1}{2}},$$

$$|||u|||=\sup_{x\in D}|u(x)|,\quad |||u|||_S=\sup_{x\in S}|u(x)|.$$

THEOREM 3. *The solution $u(x)$ of the problem* (1.1) *satisfies*:

$$\|u\|\leqq M_1\|f\|+(M_1M_2)^{\frac{1}{2}}\|\varphi\|_S, \tag{2.6}$$

$$\|u\|_S\leqq (M_1M_2)^{\frac{1}{2}}\|f\|+M_2\|\varphi\|_S, \tag{2.7}$$

$$\|\nabla u\|\leqq M_1^{\frac{1}{2}}\|f\|+M_2^{\frac{1}{2}}\|\varphi\|_S \tag{2.8}$$

and

$$|||u|||\leqq M_1|||f|||+M_2|||\varphi|||_S. \tag{2.9}$$

§3. **Proofs of Theorems.** It is well known that there exists a system of eigenvalues and eigenfunctions $\{\lambda_n, \psi_n(x);\ n=0,1,2,\cdots\}$ of the operator Λ associated with the boundary condition $\partial\psi/\partial\boldsymbol{n}=0$, with the following properties:

(3.1) $\Delta\psi_n=-\lambda_n\psi_n$ in D and $\dfrac{\partial\psi_n}{\partial n}=0$ on S $(n=0,1,2,\cdots)$,

(3.2) $\lambda_0=0<\lambda_1\leqq\lambda_2\leqq\cdots\leqq\lambda_n\leqq\cdots,\ \lim_{n\to\infty}\lambda_n=\infty,$

(3.3) $\psi_0(x)\equiv|D|^{-\frac{1}{2}}$

and

(3.4) $\{\psi_n\}$ is a complete orthonormal system in $L^2(D)$;

furthermore, the fundamental solution $U(t,x,y)$ of (1.2) is non-negative and expressible by

(3.5) $$U(t,x,y)=\sum_{n=0}^{\infty}e^{-\lambda_n t}\psi_n(x)\psi_n(y)$$

(see, for example, Theorem 10 in [2; p. 89]).

Proof of Theorem 1. It may be seen from arguments in [2] that

(3.6) $$\int_D U(t,x,y)\,dy=1 \text{ and } \int_S U(t,x,y)\,dS_y\leqq Mt^{-\frac{1}{2}}$$

for a suitable constant M independent of $x\in D$, and that

(3.7) $$\sup_{t>0,\ r(x,y)>\varepsilon} U(t,x,y)<\infty \quad \text{for any } \varepsilon>0$$

where $r(x,y)$ denotes the Riemannian distance between x and y. On the other hand, it follows from (3.3), (3.5) and Schwarz's inequality that

(3.8) $$|U(t,x,y)-|D|^{-1}|^2\leqq|U(t,x,x)-|D|^{-1}|\cdot|U(t,y,y)-|D|^{-1}|$$

and

(3.9) $$\begin{aligned}0\leqq U(t,x,x)-|D|^{-1}&\leqq e^{-\lambda_1(t-1)}\sum_{n=1}^{\infty}e^{-\lambda_n}\psi_n(x)^2\\ &\leqq e^{-\lambda_1(t-1)}\{U(1,x,x)-|D|^{-1}\} \quad \text{for} \quad t>1.\end{aligned}$$

Hence we have

(3.10) $$\begin{aligned}&|U(t,x,y)-|D|^{-1}|\\ &\leqq e^{-\lambda_1(t-1)}\{U(1,x,x)-|D|^{-1}\}^{\frac{1}{2}}\{U(1,y,y)-|D|^{-1}\}^{\frac{1}{2}} \quad \text{for} \quad t>1.\end{aligned}$$

Combining this inequality with (3.6) and (3.7), we may see that $K(x,y)$ in (2.2) is well defined if $x,y\in\overline{D}$ and $x\neq y$, and that M_1 and M_2 defined in (2.4) are finite. (2.3) follows from (3.5) and (2.2). Theorem 1 is thus proved.

Proof of Theorem 2. Case 1: $\varphi\equiv0$. If such is the case, then (2.1) means $(f,\psi_0)=0$. Hence we have

$$f(x)=\sum_{n=1}^{\infty}\alpha_n\psi_n(x) \text{ (mean convergence) where } \alpha_n=(f,\psi_n).$$

Therefore

$$\int_D U(t,x,y)\,f(y)\,dy=\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n t}\psi_n(x) \qquad \text{(by (3.5))}$$

and hence, if we define $u(x)$ by (2.5) (with $\varphi\equiv0$), we have

$$u(x)=-\int_0^{\infty}\sum_{n=1}^{\infty}\alpha_n e^{-\lambda_n t}\psi_n(x)\,dt=-\sum_{n=1}^{\infty}\frac{\alpha_n}{\lambda_n}\psi_n(x) \quad \text{(mean convergence)}$$

by means of (2.2). Hence, for any function $h(x)\in C^2(D)\cap C^1(\overline{D})$ satisfying

$$\int_D |\Lambda h(x)|^2 dx < \infty \quad \text{and} \quad \frac{\partial h}{\partial \boldsymbol{n}} = 0 \text{ on } S,$$

it holds that

$$(\Lambda h, u) = -\sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda_n} (\Lambda h, \psi_n) = \sum_{n=1}^{\infty} \alpha_n (h, \psi_n) = (h, f).$$

Therefore, by virtue of properties of $U(t, x, y)$ stated in [2], we get

$$\frac{\partial}{\partial t} \int_D U(t, x, y)\, u(y)\, dy = \int_D \Lambda_y\, U(t, x, y)\, u(y)\, dy \tag{3.11}$$
$$= \int_D U(t, x, y)\, f(y)\, dy$$

(the subscript y to Λ means to operate Λ to $U(t, x, y)$ as a function of y) and accordingly

$$\int_D U(t, x, y)\, u(y)\, dy - u(x) = \int_0^t d\tau \int_D U(\tau, x, y)\, f(y)\, dy. \tag{3.12}$$

Hence $u(x)$ satisfies $\partial u/\partial \boldsymbol{n} = 0$ on S and $\partial u/\partial t = \Lambda u - f$ $(x \in D)$; the latter equation means $\Lambda u = f$. Thus Theorem 2 is proved in case $\varphi \equiv 0$.

Case 2: *φ is of class C^2 on S and all partial derivatives of φ of the second order considered on S are Hölder-continuous.* In this case, we may construct a function $w(x)$ of class C^2 on $\bar{D}$ such that $\partial w/\partial \boldsymbol{n} = \varphi$ on S and that all partial derivatives of w of the second order are Hölder-continuous on $\bar{D}$. Then

$$\int_D \Lambda w(x)\, dx = \int_S \varphi(x)\, dS_x, \tag{3.13}$$

and hence $f - \Lambda w$ is Hölder-continuous in D and satisfies

$$\int_D |f(x) - \Lambda w(x)|^2 dx < \infty \quad \text{and} \quad \int_D \{f(x) - \Lambda w(x)\}\, dx = 0.$$

Hence the function $v(x)$ defined by

$$v(x) = -\int_D K(x, y) \{f(x) - \Lambda w(y)\}\, dy,$$

satisfies $\Lambda v = f - \Lambda w$ in D and $\partial v/\partial \boldsymbol{n} = 0$ on S as proved above. On the other hand, by similar computations to those in (3.11) and (3.12), we may obtain

$$\int_D U(t, x, y)\, w(y)\, dy - w(x)$$
$$= \int_0^t \left\{ \int_D U(\tau, x, y) \cdot \Lambda w(y)\, dy - \int_S U(\tau, x, y)\, \varphi(y)\, dS_y \right\} d\tau$$

for any $t > 0$. Hence, by (3.13),

$$\int_D \{U(t, x, y) - |D|^{-1}\}\, w(y)\, dy + |D|^{-1} \int_D w(y)\, dy - w(x)$$
$$= \int_0^t \left[\int_D \{U(\tau, x, y,) - |D|^{-1}\} \Lambda w(y)\, dy - \int_S \{U(\tau, x, y) - |D|^{-1}\}\, \varphi(y)\, dS_y \right] d\tau.$$

Letting $t\to\infty$, and using (2.2), (3.6), (3.9) and (3.10), we obtain

$$w(x)-c=-\int_D K(x,y)\cdot \Delta w(y)\,dy+\int_S K(x,y)\,\varphi(y)\,dS_y$$

where $c=|D|^{-1}\int_D w(y)\,dy$. Hence the function $u(x)$ defined by (2.5) is equal to $v(x)+w(x)-c$ which clearly satisfies (1.1).

Case 3: *General case.* We may construct a sequence $\{\varphi_n(x)\}$ of functions on S such that each $\varphi_n(x)$ satisfies the assumption for $\varphi(x)$ in Case 2 and that $\lim_{n\to\infty}\varphi_n(x)=\varphi(x)$ uniformly on S. We put

$$c_n=\int_S \varphi_n(x)\,dS_x-\int_S \varphi(x)\,dS_x \quad\text{and}\quad f_n(x)=f(x)+c_n.$$

Then we have

$$\int_D f_n(x)\,dx=\int_S \varphi_n(x)\,dS_x \text{ and } \lim_{n\to\infty} f_n(x)=f(x) \text{ uniformly in } D.$$

Hence (see Case 2) the function

$$u_n(x)=-\int_D K(x,y)\,f_n(y)\,dy+\int_S K(x,y)\,\varphi_n(y)\,dS_y \tag{3.14}$$

satisfies $\Delta u_n=f_n$ in D and $\partial u_n/\partial \boldsymbol{n}=\varphi_n$ on S, and hence, by similar computations to (3.11–12), we get

$$\begin{aligned} u_n(x)=&\int_D U(t,x,y)\,u_n(y)\,dy \\ &-\int_0^t\left\{\int_D U(\tau,x,y)\,f_n(y)\,dy-\int_S U(\tau,x,y)\,\varphi_n(y)\,dS_y\right\}d\tau \end{aligned} \tag{3.15}$$

for any $t>0$. On the other hand, $u_n(x)$ defined by (3.14) converges to $u(x)$ defined by (2.5), as $n\to\infty$, uniformly on $\overline{D}$. Hence, letting $n\to\infty$ in (3.15), we obtain

$$\begin{aligned} u(x)=&\int_D U(t,x,y)\,u(y)\,dy \\ &-\int_0^t\left\{\int_D U(\tau,x,y)\,f(y)\,dy-\int_S U(\tau,x,y)\,\varphi(y)\,dS_y\right\}d\tau. \end{aligned}$$

Hence, by means of the result of [2], $u(x)$ satisfies $\partial u/\partial \boldsymbol{n}=\varphi$ on S and $\partial u/\partial t=\Delta u-f$ which means $\Delta u=f$. Proof of Theorem 2 is thus complete.

Proof of Theorem 3. By means of the uniqueness of solution, $u(x)$ is expressible by (2.5). Hence, if we put

$$u_1(x)=-\int_D K(x,y)\,f(y)\,dy \quad\text{and}\quad u_2(x)=\int_S K(x,y)\,\varphi(y)\,dS_y.$$

we have

$$u(x)=u_1(x)+u_2(x). \tag{3.16}$$

By means of Schwarz's inequality and (2.4), we get

$$|u_1(x)|^2 \leqq \int_D |K(x,y)|\,dy \int_D |K(x,y)|\cdot|f(y)|^2\,dy \leqq M_1 \int_D |K(x,y)|\cdot|f(y)|^2\,dy.$$

Hence by virtue of (2.3), (2.4) and Fubini's theorem, we obtain

$$(3.17) \qquad \|u_1\|^2 \leqq M_1 \int_D \int_D |K(x,y)|\cdot|f(y)|^2\,dy\,dx \leqq M_1^2 \|f\|^2.$$

Similarly we may show that

$$(3.18) \qquad \|u_1\|_S^2 \leqq M_1 M_2 \|f\|^2,$$

$$(3.19) \qquad \|u_2\|^2 \leqq M_1 M_2 \|\varphi\|_S^2$$

and

$$(3.20) \qquad \|u_2\|_S^2 \leqq M_2^2 \|\varphi\|_S^2.$$

Hence, by (3.16), we obtain (2.6) and (2.7). (2.8) is proved as follows. By means of Green's formula and Theorem 2, we have

$$\|\nabla u_1\|^2 = -(\Delta u_1, u_1) = -(f, u_1) \leqq \|f\|\cdot\|u_1\|$$

and

$$\|\nabla u_2\|^2 = \int_S \frac{\partial u_2(x)}{\partial n} u_2(x)\,dS_x \leqq \|\varphi\|_S \cdot \|u_2\|_S.$$

Accordingly, by (3.17) and (3.20), we get $\|\nabla u_1\| \leqq M_1^{\frac{1}{2}} \|f\|$ and $\|\nabla u_2\| \leqq M_2^{\frac{1}{2}} \|\varphi\|_S$, which imply (2.8). Finally (2.9) is obvious from (2.4) and (2.5).

References

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[2] S. Itô: Fundamental solutions of parabolic differential equations and boundary value problems, Jap. J. Math., **27**, 55–102 (1957).