113. Note on Finitely Generated Projective Modules

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It is well known that any projective module P over a local ring is free and that if P is finitely generated any generating set for P contains a basis.[1] Recentely M. Chadeyras proved in [3] that any finitely generated projective module over a commutative quasi-semilocal[2] integral domain is free. In this note we shall generalize his theorem to finitely generated projective modules over an indecomposable quasi-semi-local commutative ring.

Every ring considered in this note is commutative and has a unit element which acts as unit operator on any module.

Let S be a multiplicatively closed set not containing 0 of a (commutative) ring R. Then as usual we denote by R_S the ring of quotient with respect to S, and if $S=R-\mathfrak{p}$ for a prime ideal $\mathfrak{p}$, we write $R_{\mathfrak{p}}$ for $R_{R-\mathfrak{p}}$. Let M be an R-module, then the relation of couples (m_1, s_1), (m_2, s_2) where $m_i \in M$, $s_i \in S$:

"there exists $s \in S$ such that $s(s_1m_2-s_2m_1)=0$" is an equivalence relation. We denote by M_S the set of equivalence classes. Then M_S may be considered as an R_S-module and $M_S=M\otimes_R R_S$. Further exists a natural map $\varphi: M \rightarrow M_S$ and the kernel of this map is the S-component of 0 in M: $\operatorname{Ker}\varphi=\{m \in M \mid \text{there exists } s \in S \text{ such that } sm=0\}$.

LEMMA 1. *If P is a finitely generated projective module over a (commutative) ring R. Then $(0:P)$ is a direct summand of R.*

For the proof, we refer the reader to [4].

DEFINITION. *A ring R is called indecomposable if there exist no proper ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $R=\mathfrak{a}\oplus\mathfrak{b}$.*

COROLLARY. *Any finitely generated projective module $P(\neq 0)$ over an indecomposable ring R is faithful* (i.e., $(0:P)=0$).

The following lemma is well-known.

LEMMA 2. *Let P be a projective module with a set of generators $(p_1, p_2, \cdots)$. Then there exist a free module F with a basis $(u_1, u_2, \cdots)$, and a submodule Q of F such that $F=P\oplus Q$ and $u_i=p_i+q_i$, $q_i \in Q$.*

LEMMA 3. *Let F be a finitely generated free module over an indecomposable ring R, and let P, Q be submodules of F such that $F=P\oplus Q$. Then P is not contained in $\mathfrak{m}F$ for any maximal ideal*

1) Cf. [1]. See also [5] for the infinite case.

2) A ring is called quasi-semi-local if it has only a finite number of maximal ideals.

$\mathfrak{m}$ *of* R.

PROOF. If P is contained in $\mathfrak{m}F$, we have $P=\mathfrak{m}P$. For: let p be any element of P, then $p=\sum m_i u_i$, $m_i \in \mathfrak{m}$, $u_i \in F$. If $u_i=p_i+q_i$, $p_i \in P, q_i \in Q$, then $p=\sum m_i p_i$ is contained in $\mathfrak{m}P$. Thus we have $\mathfrak{m}R_{\mathfrak{m}}P_{\mathfrak{m}}=P_{\mathfrak{m}}$. From this we have, by the well-known Nakayama's lemma, $P_{\mathfrak{m}}=0$. Therefore P is the $(R-\mathfrak{m})$-component of 0 in P, i.e., there exists an element $s \in (R-\mathfrak{m})$ such that $sp=0$ for any element p of P. Since P is finitely generated there exists an element $s' \in (R-\mathfrak{m})$ such that $s'P=0$ and this is a contradiction since P is projective hence faithful.

PROPOSITION 1. *Let P be a finitely generated projective module over an indecomposable quasi-semi-local ring R with only two maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Then P is free and the Z-module $\bar{Z}=Z(x_1,\cdots,x_m)$ generated, over the ring of integers Z, by a generating set $(x_1,\cdots,x_m)$ of P over R contains a free basis.*

PROOF. Suppose that $p_1,\cdots,p_n$ generate P (over R) $p_i \in \bar{Z}$, and that any $n-1$ elements of $\bar{Z}$ do not generate P. We shall prove that $(p_1,\cdots,p_n)$ is a free basis for P over R. Let F be a free module with a free basis $(u_1,\cdots,u_n)$ and Q a submodule of F such that $F=P\oplus Q$, $u_i=p_i+q_i$, $q_i \in Q$. If $Q\neq 0$, there exists, by Lemma 3, an element $q_1'=a_1u_1+\cdots+a_nu_n$ of Q such that $a_s \notin \mathfrak{m}_1$ for at least one index s, $1\leq s\leq n$. Similary there exists an element $q_2'=b_1u_1+\cdots+b_nu_n$ of Q such that $a_{s'} \notin \mathfrak{m}_2$ for at least one index s', $1\leq s'\leq n$. Let e_1 and e_2 be elements of R such that $e_1 \notin \mathfrak{m}_1$, $e_1 \in \mathfrak{m}_2$, $e_2 \in \mathfrak{m}_1$, $e_2 \notin \mathfrak{m}_2$. Now put $q'=e_1q_1'+e_2q_2'=c_1u_1+\cdots+c_nu_n$. Then we have $c_s \notin \mathfrak{m}_1$ and $c_{s'} \notin \mathfrak{m}_2$. If $s=s'$, $(u_1,\cdots,u_{s-1},q',u_{s+1},\cdots,u_n)$ is a free basis of F, since c_s is inversible in R. Let p be any element of P and $p=r_1u_1+\cdots+r_{s-1}u_{s-1}+r_sq'+r_{s+1}u_{s+1}+\cdots+r_nu_n$. Then we have that $p=r_1p_1+\cdots+r_{s-1}p_{s-1}+r_{s+1}p_{s+1}+\cdots+r_np_n$. Therefore P is generated by $n-1$ elements $p_1,\cdots,p_{s-1},p_{s+1},\cdots,p_n$ of $\bar{Z}$, and this is a contradiction. Therefore we may assume that $s<s'$ and that $c_s \notin \mathfrak{m}_1$, $c_s \in \mathfrak{m}_2$, $c_{s'} \in \mathfrak{m}_2$, $c_{s'} \notin \mathfrak{m}_2$. Then we have

$$q'=c_1u_1+\cdots+c_{s-1}u_{s-1}+(c_s+c_{s'})u_s+c_{s+1}u_{s+1}+\cdots+c_{s'-1}u_{s'-1}+c_{s'}(u_{s'}-u_s)+c_{s'+1}u_{s'+1}+\cdots+c_nu_n.$$

Now $(u_1,\cdots,u_{s'-1},(u_{s'}-u_s),u_{s'+1},\cdots,u_n)$ and $(u_1,\cdots,u_{s-1},q',u_{s+1},\cdots,u_{s'-1},(u_{s'}-u_s),u_{s'+1},\cdots,u_n)$ are free bases for F, since $c_s+c_{s'}$ is inversible in R. Therefore P is generated by $n-1$ elements $p_1,\cdots,p_{s-1},p_{s+1},\cdots,p_{s'-1},(p_{s'}-p_s),p_{s'+1},\cdots,p_n$ of $\bar{Z}$. This contradiction proves that $Q=0$, and $(p_1,\cdots,p_n)$ is a free basis of P.

PROPOSITION 2. *Let R be a quasi-semilocal indecomposable ring, $\mathfrak{m}_1,\cdots,\mathfrak{m}_n$ the set of maximal ideals and let P be a finitely generated projective module, $(x_1,\cdots,x_m)$ a set of generators of P. Then P is* *free. Furthermore, if we take n elements* $e_1,\cdots,e_n$ *of R such that* $e_i\in\mathfrak{m}_1\frown\cdots\frown\mathfrak{m}_{i-1}\frown\mathfrak{m}_{i+1}\frown\cdots\frown\mathfrak{m}_n$, $e_i\notin\mathfrak{m}_i$, *and put* $\overline{Z}=Z[e_1,\cdots,e_n](x_1,\cdots,x_m)$ *(a submodule of P generated by* $(x_1,\cdots,x_m)$ *over a subring* $Z[e_1,\cdots,e_n]$ *of R), then* $\overline{Z}$ *contains a free basis of P.*

PROOF. Let $p_1,\cdots,p_t$ be elements of $\overline{Z}$ such that $p_1,\cdots,p_t$ generate P over R, but any $t-1$ elements of $\overline{Z}$ do not generate P. We shall prove that $(P_1,\cdots,P_t)$ is a free basis of P. Let F be a free module with a basis $(u_1,\cdots,u_t)$ and Q a submodule of F such that $F=P\oplus Q$, $u_i=p_i+q_i$, $q_i\in Q$. Then it suffices to prove that $Q=0$. Suppose that $Q\neq 0$, and there exist, by Lemma 3, elements $q_1',\cdots,q_n'$ of Q such that $q_i'\notin\mathfrak{m}_iF$. Put $q'=e_1q_1'+\cdots+e_nq_n'=a_1u_1+\cdots+a_tu_t$. Then $q'\notin\mathfrak{m}_iF$ for $i=1,2,\cdots,n$. Let $\mathfrak{S}_i$ be the set of indices j such that $a_i\notin\mathfrak{m}_j$. Then $\bigcup_i\mathfrak{S}_i=\{1,2,\cdots,n\}$. Put $\overline{\mathfrak{S}}_i=\mathfrak{S}_i-\bigcup_{j=1}^{i-1}\mathfrak{S}_j=\{i(1),\cdots,i(s(i))\}$, and $e_i'=e_{i(1)}+\cdots+e_{i(s(i))}e_j'=0$ if $\overline{\mathfrak{S}}=\phi$. Then $\bigcup_{i=1}^{t}\overline{\mathfrak{S}}_i=\{1,\cdots,n\}$ and $\overline{\mathfrak{S}}_i\frown\overline{\mathfrak{S}}_j=\phi$ if $i\neq j$. Then we have that

$$q'=(a_1+e_2'a_2+\cdots+e_t'a_t)u_1+a_2(u_2-e_2'u_1)+\cdots+a_t(u_t-e_t'u_1)$$

and that $a_1+e_2'a_2+\cdots+e_t'a_t$ is not contained in $\mathfrak{m}_i$ for $i=1,\cdots,n$. Thus $(u_1,u_2-e_2'u_1,\cdots,u_t-e_t'u_1)$ and $(q',u_2-e_2'u_1,\cdots,u_t-e_t'u_1)$ are free bases of F over R, and therefore P is generated by $t-1$ elements $p_2-e_2'p_1,\cdots,p_t-e_t'p_t$ of $\overline{Z}$. This contradiction completes the proof.

PPOPOSITION 3. *Let M be a finitely generated module over a quasi-semilocal ring R. If, for each maximal ideal* $\mathfrak{m}$ *of R, the module* $M_\mathfrak{m}$ *over the ring* $R_\mathfrak{m}$ *is free, then M is R-projective.*[3)]

PROOF. Let $(m_1,\cdots,m_n)$ be a set of generators of M, F a free module with a basis $(u_1,\cdots,u_n)$. Let φ be an R-homomorphism of F onto M such that $\varphi(u_i)=m_i$, and K the kernel of φ. Then we have an exact sequence $0\to K\xrightarrow{\psi}F\xrightarrow{\varphi}M\to 0$, where ψ is the inclusion map. Now let $\mathfrak{m}_1,\cdots,\mathfrak{m}_t$ be the set of maximal ideals. Then, since $\otimes_R R_{\mathfrak{m}_j}$ is an exact functor, we have that, for each j, the sequence $0\to K_j\xrightarrow{\psi_j}F_j\xrightarrow{\varphi_j}M_j\to 0$ is exact where we put $N_j=N_{\mathfrak{m}_j}$ for any module N. By hypothesis M_j is projective. Therefore this exact sequence splits, whence there exists a homomorphism $\Psi_j\colon F_j\to K_j$ such that $1=\Psi_j\psi_j$. First, we shall prove that K is finitely generated. We recall that, if K_j' is the $(R-\mathfrak{m}_j)$-component of 0 in K, K/K_j' is an R-submodule of K_j. Now, if we denote by u_{ij} the image of u_i by the natural map of F into F_j, there exist elements $s_j\in R-\mathfrak{m}_j$ such that $s_j\Psi_j(u_{ij})\in K/K_j'$ for each i and j. Let k_{ij} be representatives of $s_j\Psi_j(u_{ij})$ mod. K_j', and Ψ_j' a homomorphism of F into K such that $\Psi_j'(u_i)=k_{ij}$. Let $\overline{K}_j$ be the submodule of K generated by the set

3) Cf. Lemma 5 (p. 249) of [2].

$(k_{1j}, \cdots, k_{nj})$. Then K is the $(R-\mathfrak{m}_j)$-component of $\overline{K}_j$ in K. From this, we have that K is generated by the set $\{k_{ij}\}$ where $i=1, \cdots, n$; $j=1, \cdots, t$. For: let $\overline{K}$ be the submodule generated by the set $\{k_{ij}\}$, k any element of K. Then $\{r \in R \mid rk \in \overline{K}\} = (\overline{K} : k) = R$, whenec $k \in \overline{K}$. Now let $k_1, \cdots, k_m$ be a set of generators of $\overline{K}$. Then we have $(\Psi'_j \psi k_i) - s_j k_i \in K'_j$, thus there exists an element $s'_j \in (R-\mathfrak{m}_j)$ such that $((s'_j \Psi'_j)\psi k_i) - s'_j s_j k_i = 0$ for $i=1, \cdots, m$. This implies that $(\mathrm{Hom}_R(F, K)) \circ \psi$ contains the identity map of $\mathrm{Hom}_R(K, K)$. Therefore the exact sequence $0 \to K \to F \to M \to 0$ splits and thus M is projective.

Added in Proof: Prof. S. Endo has obtained the same results as our Prop. 2 and 3.

References

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