108. *On Information in Operator Algebras*

By Hisaharu UMEGAKI
Tokyo Institute of Technology

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1. In the present paper, we shall introduce a non-commutative information in an operator algebra. This may be useful for the theory of entropy in quantum statistics (cf. Nakamura-Umegaki [5]).

Let A be a von Neumann algebra with a faithful normal trace τ, and $L^p=L^p(A)=L^p(A,\tau)$ $(p\geqq 1)$ be Banach space consisting of all measurable operators a with the finite integral $\tau(|a|^p)<+\infty$, where the norm is defined by $\|a\|_p=(\tau(|a|^p))^{1/p}$ (cf. Dixmier [1], Segal [6]).

Let S be a set of all normal states $\sigma, \rho, \cdots$ of A. For any $\sigma\in S$, there exists uniquely an operator $d(\sigma)\in L^1$ such that

$$\sigma(a)=\tau(d(\sigma)a) \quad \text{for every } a\in A.$$

The operator $d(\sigma)$ is so-called Radon-Nikodym derivative (of σ with respect to τ), this is due to Dye [2].

For the real valued function $h(\lambda)$ $(\lambda\geqq 0)$ such that

$$(1) \qquad h(\lambda)=-\lambda\log\lambda \ (\lambda>0), \ =0 \ (\lambda=0),$$

an operator function $h(a)$ is defined by

$$h(a)=\int_0^\infty h(\lambda)dE_\lambda$$

for $a\in L^1$ with the spectral resolution $a=\int_0^\infty \lambda dE_\lambda$. Denote

$$(2) \qquad H(a)=\tau(h(a))$$

and it is called *entropy* of the operator a (cf. Nakamura-Umegaki [4]). For any $\sigma\in S$, the entropy $H(d(\sigma))$ of $d(\sigma)$ is denoted by $H(\sigma)$ and it is called the *entropy* of the state σ (cf. Segal [7]).

Segal [7] has proved that the function $H(\sigma)$ over S is concave, and Nakamura-Umegaki [4] has generalized it such as the operator function $h(a)$ over $\{a\in A;\ a\geqq 0\}$ is concave, i.e.

$$(3) \qquad h(\alpha a+\beta b)\geqq \alpha h(a)+\beta h(b)$$

for every $a, b\in A$, $a, b\geqq 0$ and $\alpha, \beta\geqq 0$, $\alpha+\beta=1$. The inequality (3) is extended to the operators $a, b\in L^1$, $a, b\geqq 0$. The entropy $H(a)$ of $a\geqq 0$ is uniquely determined as $-\infty\leqq H(a)\leqq 1$ by a and the trace τ. While, the entropy $H(\sigma)$ of $\sigma\in S$ is determined only by σ and independent from the choice of τ.

2. In the theory of information, various methods have been introduced and discussed by several authors. In the present case we shall introduce into the von Neumann algebra A the amount of information of Kullback-Leibler [3].

Definition 1. For any pair $a, b \in L^1$, $a, b \geqq 0$ with same supporting projection and satisfying $\tau(a)=\tau(b)=1$,

$$I(a, b)=\tau(a \log a - a \log b)$$

is said to be the *information* between a and b.

In the following cases, the information $I(a, b)$ is uniquely determined as finite or $+\infty$:

(i) $ab=ba$,

(ii) *the entropy* $H(a)$ *is finite and* b *is bounded.*

Indeed, (i) is the case of usual probability space as the following: Let p be the supporting projection of a and b, then the operators $(b+p)^{-1}$ and $a(b+p)^{-1}$ are defined as measurable and $\geqq 0$, and hence

$$a \log a - a \log b = a(b+p)^{-1}(\log a(b+p)^{-1})b$$

is a measurable operator. Furthermore

$$a(b+p)^{-1} \log (a(b+p)^{-1}) \geqq -1 \text{ and } \tau(b)=1$$

imply that

$$I(a, b)=\tau([a(b+p)^{-1}(\log (a(b+p)^{-1})b])$$

is well-defined as finite or $+\infty$. While (ii) implies $a \log a \in L^1$ and $\tau(a \log b) \leqq \| b \|_\infty$ ($\| \ \|_\infty$ being operator bound) and hence $I(a, b) = \tau(a \log b) - \tau(a \log b)$ is well-defined as finite or $+\infty$.

In general, we may show that

THEOREM 1. *For any pair* $a, b \in L^1$, $a, b \geqq 0$ *with same supporting projection and with finite entropies* $H(a)$ *and* $H(b)$, *the information* $I(a, b)$ *is uniquely determined as finite or* $+\infty$.

As in the probability space, we can introduce the concept of divergence:

Definition 2. For any pair a, b as in Definition 1,

$$J(a, b)=I(a, b)+I(b, a)$$

is said to be the *divergence* between a and b.

The function $J(\ ,\ )$ has the separated property:

THEOREM 2. *For the pair of operators* a, b *given in Theorem* 1, *the following conditions are equivalent each other*: (i) $a=b$, (ii) $ab=ba$ *and* $I(a, b)=0$, *and* (iii) $J(a, b)=0$.

The information and the divergence between a pair of states $\sigma, \rho \in S$, which are absolutely continuous with respect to each other, are defined by the same way of the pair of operators a, b:

$$I(\sigma, \rho)=\tau([d(\sigma)\log d(\sigma) - d(\sigma)\log d(\rho)])$$

and

$$J(\sigma, \rho)=I(\sigma, \rho)+I(\rho, \sigma)$$

respectively. Then we may obtain that:

THEOREM 3. *Let* $\sigma, \rho \in S$ *be the pair given above. Suppose the entropies* $H(\sigma)$ *and* $H(\rho)$ *are finite, then the information* $I(\sigma, \rho)$ *and the divergence* $J(\sigma, \rho)$ *are uniquely determined as finite or* $+\infty$. *These are dependent only on* σ, ρ *and independent from the choice* *of the trace* τ.

Kullback-Leibler [3] have proved that in probability space the amounts of the information $I(\sigma, \rho)$ and the divergence $J(\sigma, \rho)$ are additive for independent random events. This theorem may be extended to the von Neumann algebra A:

THEOREM 4. *If* A *is direct product* $A_1 \otimes A_2$ *of von Neumann algebras* A_1 *and* A_2, *then for any* $a_i, b_i \in L^1(A_i), a_i, b_i \geqq 0$ $(i=1, 2)$ *with finite entropies and with same supporting projection*

$$I(a_1 \otimes a_2, b_1 \otimes b_2) = I(a_1, b_1) + I(a_2, b_2)$$

and

$$J(a_1 \otimes a_2, b_1 \otimes b_2) = J(a_1, b_1) + J(a_2, b_2).$$

These results will be applied to a characterization of sufficient von Neumann subalgebra of A.

The detailed proofs of the theorems stated in the present paper will be given in following our paper with their allied topics.

References

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