104. On Ascoli Theorems

By R. W. BAGLEY
University of Alabama

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Irving Glicksberg [2, Theorem 2] proved that every continuous real-valued function on a topological space X is bounded[1] if and only if every bounded equicontinuous family of functions in $C^*(X, R)$ has compact closure in the uniform topology on $C^*(X, R)$, the space of bounded continuous functions on X to the real numbers R. In this paper we obtain results related to this and other Ascoli type theorems.

Our terminology involving uniform spaces and topologies on functions spaces follows closely that of [3].

If X is a topological space, $(Y, \mathcal{V})$ is a uniform space and $\{f_n\}$ is a sequence in $C(X, Y)$, the set of continuous functions on X to Y, then $\{f_n\}$ is said to converge uniformly at x to f if, for $V \in \mathcal{V}$ there is a neighborhood U_x and an integer N such that $(f_n(y), f(y)) \in V$ whenever $n \geq N$ and $y \in U_x$. $\{f_n\}$ is said to be uniformly Cauchy at x if, for $V \in \mathcal{V}$ there is a neighborhood U_x and an integer N such that $(f_n(y), f_m(y)) \in V$ whenever $n, m \geq N$ and $y \in U_x$.

Lemma 1. If $(Y, \mathcal{V})$ is a uniform space and $\{f_n\}$ is a sequence in $C(X, Y)$ which is uniformly Cauchy at each point of X and converges pointwise to a function f, then f is continuous and $\{f_n\}$ converges uniformly at each point of X.

Proof. The pointwise uniform convergence follows as a special case of Theorem 10(b), page 229 [3]. The continuity of f follows easily.

Lemma 2. If X is pseudo-compact, $(Y, \mathcal{V})$ is a uniform space and $\{f_n\}$ is a sequence in $C(X, Y)$ which converges uniformly at each point of X to a function f, then $\{f_n\}$ converges uniformly on X to f.

Proof. If $\{f_n\}$ does not converge uniformly to f, then there is a sequence $\{x_j\}$ in X, a subsequence $\{f_{n_j}\}$, a positive number r and a pseudo-metric p in the gage of $\mathcal{V}$ such that $p(f_{n_j}(x_j), f(x_j)) > r$ for each j. We let

$$g_j(x) = \max(p(f_{n_j}(x), f(x)) - r, 0).$$

For each point x in X there is a neighborhood of x on which all except finitely many of the functions g_j vanish. Thus we can define the following continuous function

1) If every continuous real valued function on X is bounded, we say that X is pseudo-compact.

$$g(x)=\sum_{j=1}^{\infty}\frac{j}{g_j(x_j)}\,g_j(x).$$

We note that $g(x_j)\geq j$. Thus g is unbounded contradicting the hypothesis that X is pseudo-compact. It follows that $\{f_n\}$ is uniformly convergent.

Lemma 3. If X is pseudo-compact, $(Y, \mathcal{V})$ is a uniform space and $\{f_n\}$ is a sequence in $C(X, Y)$ which is uniformly Cauchy at each point of X, then $\{f_n\}$ is uniformly Cauchy on X.

Proof. This lemma follows from Lemma 2 and the fact that every uniform space is uniformly isomorphic to a dense subspace of a complete uniform space.

Theorem 1. If X is pseudo-compact, $(Y, \mathcal{V})$ is a uniform space, $\{f_n\}$ is an equicontinuous sequence in $C(X, Y)$ and $\{f_n(b)\}$ is Cauchy for each b in a dense subset B of X, then $\{f_n\}$ is uniformly Cauchy on X.

Proof. If $x\in X$ and $V\in\mathcal{V}$, then there is a neighborhood U_x and an integer N such that $(f_n(y), f_n(z))\in V_1$ whenever $n\geq N$ and $y, z\in U_x$ for some V_1 in $\mathcal{V}$ such that $V_1\circ V_1\circ V_1\subset V$. Now, there is a point $b\in U_x\cap B$ and an integer $M\geq N$ such that $(f_n(b), f_m(b))\in V_1$ whenever $n, m\geq M$. Since $(f_n(y), f_n(b))$, $(f_n(b), f_m(b))$ and $(f_m(b), f_m(y))$ are all in V_1 we have $(f_n(y), f_m(Y))$ in V whenever $n, m\geq M$ and $y\in U_x$. Thus $\{f_n\}$ is uniformly Cauchy at x. The desired conclusion follows from Lemma 3.

Corollary. If the hypothesis of Theorem 1 is satisfied and the set $\{f_n(x)\,|\,n=1, 2,\cdots\}$ has complete closure in Y for each x in X, then $\{f_n\}$ is uniformly convergent on X.

Proof. Apply Theorem 1, Lemma 1, and Lemma 2.

Lemma 4. If X is a separable, pseudo-compact space, $(Y, \mathcal{V})$ is a uniform space and $\{f_n\}$ is an equicontinuous sequence in $C(X, Y)$ such that the set $\{f_n(x)\,|\,n=1, 2,\cdots\}$ has sequentially compact closure in Y for each x in X, then $\{f_n\}$ has a uniformly convergent subsequence.

Proof. Let B be a countable dense subset of X. By a diagonal process we obtain a subsequence which converges on B. By the corollary to Theorem 1 it follows that this subsequence is uniformly convergent on X.

The following theorem is related to a theorem of J. L. Kelley [3, page 238].

Theorem 2. If X is a separable, pseudo-compact space, $(Y, \mathcal{V})$ is a uniform space and F is an equicontinuous family of functions in $C(X, Y)$ such that $F(x)$ has sequentially compact closure in Y for each x in X, then F has sequentially compact closure in the topology of uniform convergence.

Proof. This follows from Lemma 4.

Theorem 3. If X is pseudo-compact, (Y, σ) is a metric space and F is an equicontinuous family in $C(X, Y)$ such that $F(x)$ has compact closure in Y for each x, then F has compact closure in the uniform topology.[2]

Proof. We define a pseudo-metric for X as follows:

$$\delta(a, b)=\sup_{f \in F} \sigma(f(a), f(b)).$$

The pseudo-metric topology contains the original topology for X since F is equicontinuous. Therefore (X, δ) is pseudo-compact since X with the original topology is. Thus (X, δ) is compact since it is completely regular and paracompact. (Cf. Theorem 10, page 505 [1].) Clearly F is equicontinuous as a family of functions on (X, δ) to Y. By Theorem 21, page 236 [3], the family F has compact closure in the uniform topology.

A metric space is said to be finitely compact if each closed and bounded subset is compact.

Theorem 4. If X is pseudo-compact, Y is a finitely compact metric space and F is a pointwise bounded equicontinuous family in $C(X, Y)$, then F has compact closure in the uniform topology.

Proof. Since $F(x)$ is a bounded subset of a finitely compact metric space, the theorem follows immediately from Theorem 3.

References

[1] R. W. Bagley, E. H. Connell, and J. D. McKnight: On properties characterizing pseudo-compact spaces, Proc. Amer. Math. Soc., **9**, 500–506 (1958).

[2] I. Glicksberg: The representation of functionals by integrals, Duke Math. J., **19**, 253–261 (1952).

[3] J. L. Kelley: General Topology, Van Nostrand, 1955.

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2) $C(X, Y)=C^*(X, Y)$ since X is pseudo-compact.