1. The theorem. Let $X$ be a complete locally convex linear topological space, and $L(X, X)$ the algebra of all continuous linear operators on $X$ into $X$. A pseudo-resolvent $J_z$ is a function on a subset $D(J)$ of the complex plane with values in $L(X, X)$ satisfying the resolvent equation

$$ J_z - J_y = (\mu - \lambda)J_z J_y. $$

We have, denoting by $I$ the identity operator,

$$ (I - \lambda J_z) = (I - (\lambda - \mu)J_z)(I - \mu J_y) $$

and

$$ \lambda J_z (I - \mu J_y) = (1 - \mu(\lambda - \mu)^{-1}) \lambda J_z - \lambda(\lambda - \mu)^{-1} \mu J_y. $$

We see, by (1), that all $J_z, \lambda \in D(J)$, have a common null space $N(J)$ and a common range $R(J)$. We also see, by (2), that all $(I - \lambda J_z), \lambda \in D(J)$, have a common null space $N(I - J)$ and a common range $R(I - J)$. $N(J)$ and $N(I - J)$ are closed linear subspace of $X$, but $R(J)$ and $R(I - J)$ need not be closed; we shall denote by $R(J)^c$ and $R(I - J)^c$ their closures respectively.

To formulate our ergodic theorems we prepare two lemmas.

**Lemma 1.** Let there exist a sequence $\{\lambda_n\}$ of numbers $\in D(J)$ such that

$$ \lim_{n \to \infty} \lambda_n = 0 $$

and the family of operators $\{\lambda_n J_{\lambda_n}\}$ is equi-continuous. Then we have

$$ R(I - J)^c = P(J) = \{x \in X; \lim_{n \to \infty} \lambda_n J_{\lambda_n} x = 0\}, $$

and hence

$$ N(I - J) \cap R(I - J)^c = \{0\}. $$

**Lemma 1'.** Let there exist a sequence $\{\lambda_n\}$ of numbers $\in D(J)$ such that

$$ \lim_{n \to \infty} |\lambda_n| = \infty $$

and the family of operators $\{\lambda_n J_{\lambda_n}\}$ is equi-continuous. Then we have

$$ R(J)^c = I(J) = \{x \in X; \lim_{n \to \infty} \lambda_n J_{\lambda_n} x = x\}, $$

and hence

$$ N(J) \cap R(J)^c = \{0\}. $$

Our ergodic theorems read as follows.

**Theorem 1.** Let (4) be satisfied. Let, for a given $x \in X$, there exist a subsequence $\{\lambda_n\}$ of $\{\lambda_n\}$ such that
\( (7) \) weak-lim \( \lambda_{n'} J_{n'} x = x_n \) exists.

Then \( x_h = \lim_{n \to \infty} \lambda_n J_n x \) and \( x_n \in N(I-J), x_p = x - x_h \in P(J) \).

**Corollary 1.** Let \((4)\) be satisfied, and let \( X \) be locally sequentially weakly compact. Then
\( (8) \quad X = N(I-J) \oplus R(I-J)^\circ = N(I-J) \oplus P(J) \) (direct sum).

**Theorem 1'.** Let \((4)'\) be satisfied. Let, for a given \( x \in X \), there exist a subsequence \( \{\lambda_n\} \) of \( \{\lambda_n\} \) such that
\( (7)' \) weak-lim \( \lambda_{n'} J_{n'} x = x_{n'} \) exists.

Then \( x_{n'} = \lim_{n' \to \infty} \lambda_{n'} J_{n'} x \) and \( x_{n'} \in I(J), x_{n'} = x - x_{n'} \in N(J) \).

**Corollary 1'.** Let \((4)'\) be satisfied, and let \( X \) be locally sequentially weakly compact. Then
\( (8)' \quad X = N(J) \oplus R(J)^\circ = N(J) \oplus I(J) \) (direct sum).

**Remark.** The pseudo-resolvent \( J_i \) is a resolvent of a closed linear operator \( A \) if and only if \( N(J) = 0 \); in this case \( R(J) \) coincides with the domain \( D(A) \) of \( A \). When \( J_i \) is the resolvent of a closed linear operator \( A \) with domain \( D(A) \) and range \( R(A) \) both in a Banach space \( X \), the Theorem 1 and Theorem 1' respectively correspond to the "abelian ergodic theorem at \( \infty \) for semi-groups" and the "abelian ergodic theorem at 0 for semi-groups". Our formulation is more general than those due to E. Hille and R.S. Phillips. The Theorem 1' for the case of a Banach space \( X \) is due to T. Kato, and our formulation of two theorems is modelled after him.

2. **Proofs of the theorems.**

**Proof of Lemma 1.** We see, by \((3)\) and \((4)\), that \( x \in R(I-J) \) implies \( x \in P(J) \). Let \( y \in R(I-J)^\circ \). Then, for any semi-norm \( q \) on \( X \) and \( \varepsilon > 0 \), there exists \( x \in R(I-J) \) such that \( q(y - x) < \varepsilon \). By \((4)\), we have, for any semi-norm \( q' \) on \( X \), \( q'(\lambda_n J_n (y - x)) \leq M q(y - x) \) where the positive constant \( M \) depends on \( q \) and \( q' \). Thus we see that \( y \) must belong to \( P(J) \).

Let conversely \( x \in P(J) \). Then, for any semi-norm \( q \) on \( X \) and \( \varepsilon > 0 \), there exists \( \lambda_n \) such that \( q(x - (x - \lambda_n J_n x)) < \varepsilon \). Hence \( x \) must belong to \( R(I-J)^\circ \).

The proof of Lemma 1' may be obtained similarly by making use of \((3)\) and \((4)\)'.

**Proof of Theorem 1.** Setting \( \mu = \lambda_{n'} \) in \((2)\) and letting \( n' \to \infty \), we see, by \((4)\), that \( x_n \in N(I-J) \). We have thus
\( (9) \quad \lambda_n J_n x = x_n + \lambda_n J_n (x - x_n) \),
and hence we have to prove that \( (x - x_n) \in P(J) \). We prove it from

---

1) Functional analysis and semi-groups, Providence, 502 (1957).

Lemma 1 and \((x - \lambda_n \nu J_n J_x) \in R(I - J)\), observing that, in a locally convex linear topological space \(X\), any closed linear subspace is weakly closed.

Proof of Theorem 1'. Setting \(\mu = \lambda_n\) in (3) and letting \(n' \to \infty\), we see, by (4)', that \(\lambda J_n (x - x_n') = 0\), that is, \(x_n' = x - x_n' \in N(J)\). On the other hand, we have \(x_n' \in R(J)^a\). For, \(\lambda N J_n J_x \in R(J)\) and the closed linear subspace \(R(J)^a\) is weakly closed. Thus, by Lemma 1', \(x_n' \in I(J)\).

3. Some applications.

i) An application to fractional powers of closed operators. Let \(J\) be the resolvent \((\lambda I - A)^{-1}\) of a closed linear operator \(A\) and \(D(J) = \{\lambda; \lambda > 0\}\). If we assume that

\((4)''\) the family of operators \(\{\lambda (\lambda I - A)^{-1}; \lambda > 0\}\) is equi-continuous, then the fractional power \(A^a\) of \(A\), \(0 < a < 1\), may be defined as the smallest closed extension of the operator

\[A^a x = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{a-1} (\lambda I - A)^{-1}(-Ax) d\lambda \text{ for } x \in D(A)^a.
\]

Then the equation

\[\lim_{a \to 0} A^a x = x\]

is satisfied for \(x \in D(A) \cap P(J)\). This we see from the fact that such an \(x\) satisfies the "Poisson equation"

\[\lim_{a \to 0} (\lambda I - A)^{-1}(-Ax) = x.\]

ii) An application to potential theory. Suggested by a special case when \(J\) is the resolvent of the Laplacian, we may call the elements of \(N(I - J)\) "harmonic" and the elements of \(P(J)\) "potentials." Then the Theorem 1 may be considered as an analogue of F. Riesz decomposition for subharmonic functions. The analogy is more close if we introduce the notion of "subharmonicity." To this end, we assume that a notion of "positivity", denoted by \(x \geq 0\), is defined in \(X\) in such a way that \(X\) is a semi-ordered linear space satisfying the condition:

\((13)\) a monotone increasing bounded sequence of elements \(\epsilon X\) converges weakly to an element of \(X\) which is greater than the elements of the sequence.

We further assume that \(D(J)\) contains an open interval \((0, \lambda_0)\) and the operator \(J\) is positive for \(\lambda \in (0, \lambda_0)\) in the sense that \(x \geq 0\) implies \(J_x \geq 0\). We shall call an element \(x \in X\) "subharmonic" if it satisfies

\[\lambda J_x \geq x\]

for some and hence for all \(\lambda \in (0, \lambda_0)\).

Then we can prove the following corollary of Theorem 1.

Corollary 2. Let

(15) the family of operators \( \{ \lambda J_\mu ; 0 < \lambda < \lambda_0 \} \) is equi-continuous.

Then a "subharmonic" element \( x \) is uniquely decomposed as the sum of a "harmonic" element \( x_h \) and a "potential" \( x_p \). The "harmonic" part \( x_h \) of \( x \) is given by \( x_h = \lim_{\lambda \to 0^+} \lambda J_\lambda x \) and \( x_h \) is the "least harmonic majorant" of \( x \).

Proof. By (2) and the positivity of \( J_\mu \), we see that

(16) \( \lambda < \mu \) implies \( \lambda J_\lambda x \geq \mu J_\mu x \geq x \).

Therefore, by (13) and (12), weak-lim \( \lambda J_\lambda x = x_h \) exists, and the first part of the Corollary is proved.

Let a "harmonic" element \( x_h \) satisfy \( x_h \geq x \). Then, by the positivity of \( \lambda J_\lambda \) and the "harmonicity" of \( x_h \), we have

\[ x_h = \lambda J_\lambda x \geq \lambda J_\lambda x \]

and hence \( x_h \geq x_h \).

iii) An application to semi-group theory. The following corollaries of Theorem 1' are obtained after T. Kato, loc. cit. in the footnote 2.

Corollary 2'. If \( J_\lambda \) is a pseudo-resolvent satisfying (4)' and if \( R(J) \) is dense in \( X \), then \( N(J) = \{0\} \) and hence \( J_\lambda \) is a resolvent.

Corollary 3'. If \( X \) is locally sequentially weakly compact and \( J_\lambda \) is the resolvent of a closed linear operator \( A \) satisfying (4)', then the domain \( D(A) \) of \( A \) is dense in \( X \).