133. Some Analytical Properties of the Spectra of Normal Operators in Hilbert Spaces

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Definition. Let $\mathfrak{H}$ be the complex abstract Hilbert space which is complete, separable and infinite dimensional; and let N_j, $j=1,2,\cdots,p$, be bounded normal operators in $\mathfrak{H}$. We then define a complex-valued function $S(f,g;\lambda)$ of a complex variable λ by

$$S(f,g;\lambda)=\left(\left(\sum_{j=1}^{p}\sum_{\alpha=1}^{m_j}c_{\alpha j}(\lambda I-N_j)^{-\alpha}\right)f,g\right),\ \left(f\in\bigcap_{j=1}^{p}\mathfrak{D}((\lambda I-N_j)^{-m_j}),\ g\in\mathfrak{H}\right),$$

under the assumption that the set of accumulation points of the point spectrum of each N_j is an at most denumerably infinite set.

Though f here is arbitrarily chosen in the domain $\bigcap_{j=1}^{p}\mathfrak{D}((\lambda I-N_j)^{-m_j})$ so that the function S is significant, the domains of f in the results of integrations of S along such curves as will afterwards be defined are extended respectively: because the respective integrals of $\sum_{j=1}^{p}\sum_{\alpha=1}^{m_j}c_{\alpha j}(\lambda I-N_j)^{-\alpha}$ are reduced to simplified operators as we shall see later on.

As will afterwards be verified immediately from the integral expressions and the expansions of N_j, $j=1,2,\cdots,p$, the following statements are valid:

1° $S(f,g;\lambda)$ is regular in the intersection of all resolvent sets of N_j, $j=1,2,\cdots,p$, only;

2° every isolated eigenvalue of N_j, $j=1,2,\cdots,p$, is a pole with order m_j of $S(f,g;\lambda)$;

3° though every accumulation point of isolated eigenvalues of each N_j is a non-isolated essential singularity of $S(f,g;\lambda)$ in the sense of the function theory, $S(f,g;\lambda)$ has the principal part of the expansion at that point when and only when it belongs to the point spectrum of the N_j;

4° every point belonging to the union of the continuous spectra of N_j, $j=1,2,\cdots,p$, is a non-regular point of $S(f,g;\lambda)$, but not a usual singularity in the sense of the function theory unless it is an accumulation point of isolated eigenvalues of any one of N_j, $j=1,2,\cdots,p$.

In particular, we are interested in the case where $S(f,g;\lambda)$ has a denumerably infinite set of non-isolated essential singularities. We shall discuss the integral of $S(f,g;\lambda)$ along a rectifiable closed Jordan curve comprising those denumerably infinite essential singularities inside itself.

Theorem A. Let D be a domain whose boundary ∂D is a rectifiable closed Jordan curve; let $\{z_\nu^{(j)}\}_{\nu=1,2,\cdots}$ be the point spectra of bounded normal operators $N_j, j=1,2,\cdots,p$, in $\mathfrak{H}$; let $\{z_\nu^{(j)}\}_{\substack{\nu=1,2,\cdots\\ j=1,2,\cdots,p}}$ and their accumulation points be completely contained in D; let $K_\nu^{(j)}$ be the eigenprojector of N_j corresponding to the eigenvalue $z_\nu^{(j)}$; let $S(f,g;\lambda)$ defined above be regular with respect to λ in the closure $\overline{D}$ of D except for $\{z_\nu^{(j)}\}_{\substack{\nu=1,2,\cdots\\ j=1,2,\cdots,p}}$ and their accumulation points which are denumerably infinite in number; and let $\varphi(\lambda)$ be an arbitrarily given function regular in $\overline{D}$. Then

$$(1)\qquad \frac{1}{2\pi i}\int_{\partial D}\varphi(\lambda)S(f,g;\lambda)d\lambda=\sum_{j=1}^{p}\sum_{\alpha=1}^{m_j}\sum_{\nu=1}^{\infty}\frac{c_{\alpha j}\varphi^{(\alpha-1)}(z_\nu^{(j)})}{(\alpha-1)!}(K_\nu^{(j)}f,g),$$
$$(0!=1,\ \varphi^{(0)}(z_\nu^{(j)})=\varphi(z_\nu^{(j)})),$$

where the curvilinear integration is taken in the counterclockwise direction; and moreover the series of the right-hand side converges absolutely.

Proof. Since $N_j, j=1,2,\cdots,p$, are bounded normal operators in $\mathfrak{H}$, it is first clear that there exists such a domain D as was described in the statement of the present theorem.

Now, let G be the complex z-plane, $\Delta(N_j)$ the continuous spectrum of N_j, and $\{K^{(j)}(z)\}$ the complex spectral family associated with N_j. Then we have

$$\sum_{j=1}^{p}\sum_{\alpha=1}^{m_j}c_{\alpha j}(\lambda I-N_j)^{-\alpha}=\sum_{j=1}^{p}\sum_{\alpha=1}^{m_j}\int_G\frac{c_{\alpha j}}{(\lambda-z)^\alpha}dK^{(j)}(z)\quad(\lambda\in\partial D)$$
$$=\sum_{j=1}^{p}\sum_{\alpha=1}^{m_j}\left\{\sum_{\nu=1}^{\infty}\frac{c_{\alpha j}K_\nu^{(j)}}{(\lambda-z_\nu^{(j)})^\alpha}+\int_{\Delta(N_j)}\frac{c_{\alpha j}}{(\lambda-z)^\alpha}dK^{(j)}(z)\right\}.$$

If we next denote by $z_{(j,n)}$ one of accumulation points of $\{z_\nu^{(j)}\}_{\nu=1,2,\cdots}$ such that it belongs to the continuous spectrum of N_j, then, by the hypothesis on $S(f,g;\lambda)$, every point $z\in\Delta(N_j)-\{z_{(j,n)}\}$, $j=1,2,\cdots,p$, lies outside ∂D and hence

$$\frac{1}{2\pi i}\int_{\partial D}\frac{c_{\alpha j}\varphi(\lambda)}{(\lambda-z)^\alpha}d\lambda=0,\ (z\in\Delta(N_j)-\{z_{(j,n)}\};\ j=1,2\cdots,p;\alpha=1,2,\cdots,m_j).$$

On the other hand, since, by hypotheses, $\{z_\nu^{(j)}\}_{\substack{\nu=1,2,\cdots\\ j=1,2,\cdots,p}}$ and their accumulation points lie inside ∂D, we have

$$\frac{1}{2\pi i}\int_{\partial D}\frac{c_{\alpha j}\varphi(\lambda)}{(\lambda-z_\nu^{(j)})^\alpha}d\lambda=\frac{c_{\alpha j}\varphi^{(\alpha-1)}(z_\nu^{(j)})}{(\alpha-1)!},\ (\alpha=1,2,\cdots,m_j);$$

and moreover we can derive without difficulty the relation

$$\int_{\partial D}\int_{\overline{D}}\frac{c_{\alpha j}\varphi(\lambda)}{(\lambda-z)^\alpha}dK^{(j)}(z)d\lambda=\int_{\overline{D}}\int_{\partial D}\frac{c_{\alpha j}\varphi(\lambda)}{(\lambda-z)^\alpha}d\lambda dK^{(j)}(z)$$

by considering the limit of a sequence of approximation sums of the line integral of the left-hand side.

Furthermore, if we donote by $\{E^{(j)}(\Re(z))\}$ and $\{F^{(j)}(\Im(z))\}$ the spectral families associated with the self-adjoint operators $\frac{1}{2}(N_j+N_j^*)$ and $\frac{1}{2i}(N_j-N_j^*)$ respectively and consider the rectangle δ defined by

$$(\Re(z_{(j,n)})-\varepsilon<\Re(z)\leqq\Re(z_{(j,n)}),\ \Im(z_{(j,n)})-\varepsilon'<\Im(z)\leqq\Im(z_{(j,n)}))$$

where ε and ε' are sufficiently small positive numbers, then

$$K^{(j)}(\delta)=[E^{(j)}(\Re(z_{(j,n)}))-E^{(j)}(\Re(z_{(j,n)})-\varepsilon)]$$
$$\times[F^{(j)}(\Im(z_{(j,n)}))-F^{(j)}(\Im(z_{(j,n)})-\varepsilon')]$$

and it converges in the sense of norm to the null operator as ε and ε' both tend to zero: for otherwise $z_{(j,n)}$ would become an eigenvalue of N_j, contrary to the assumption on $z_{(j,n)}$. On the other hand, the set $\{z_{(j,n)}\}$ of such points $z_{(j,n)}$, $n=1,2,\cdots$, as we described above is an at most denumerably infinite set and lies inside ∂D. Hence

$$\frac{1}{2\pi i}\int_{\partial D}\int_{\{z_{(j,n)}\}}\frac{c_{\alpha j}\varphi(\lambda)}{(\lambda-z)^\alpha}dK^{(j)}(z)d\lambda=\frac{c_{\alpha j}}{(\alpha-1)!}\int_{\{z_{(j,n)}\}}\varphi^{(\alpha-1)}(z)dK^{(j)}(z)=0$$

for $\alpha=1,2,\cdots,m_j$ and $j=1,2,\cdots,p$.

In consequence, we can find with the help of these results that

$$\frac{1}{2\pi i}\int_{\partial D}\varphi(\lambda)S(f,g;\lambda)d\lambda=\frac{1}{2\pi i}\int_{\partial D}\int_{G}\sum_{j=1}^{p}\sum_{\alpha=1}^{m_j}\frac{c_{\alpha j}\varphi(\lambda)}{(\lambda-z)^\alpha}d(K^{(j)}(z)f,g)d\lambda$$
$$=\frac{1}{2\pi i}\sum_{j=1}^{p}\sum_{\alpha=1}^{m_j}\int_{\partial D}\int_{\overline{D}}\frac{c_{\alpha j}\varphi(\lambda)}{(\lambda-z)^\alpha}d(K^{(j)}(z)f,g)d\lambda$$
$$=\sum_{j=1}^{p}\sum_{\alpha=1}^{m_j}\sum_{\nu=1}^{\infty}\frac{c_{\alpha j}\varphi^{(\alpha-1)}(z_\nu^{(j)})}{(\alpha-1)!}(K_\nu^{(j)}f,g).$$

It remains only to prove that the series in (1) converges absolutely.

Let $\mathfrak{M}^{(j)}$ be the subspace determined by all eigenelements of N_j and let $\{\varphi_k^{(j)}\}$ be an orthonormal set determining $\mathfrak{M}^{(j)}$. Then we can write

$$\left(\sum_{\nu=1}^{\infty}K_\nu^{(j)}\right)f=\sum_{k=1}^{\infty}a_k\varphi_k^{(j)},$$
$$\left(\sum_{\nu=1}^{\infty}K_\nu^{(j)}\right)g=\sum_{k=1}^{\infty}b_k\varphi_k^{(j)},$$

where $\sum_{\nu=1}^{\infty}K_\nu^{(j)}\leqq I$, $a_k=\left(\left(\sum_{\nu=1}^{\infty}K_\nu^{(j)}\right)f,\varphi_k^{(j)}\right)$ and $b_k=\left(\left(\sum_{\nu=1}^{\infty}K_\nu^{(j)}\right)g,\varphi_k^{(j)}\right)$, $k=1,2,\cdots$. On the other hand, since $\left(\sum_{\nu=1}^{\infty}K_\nu^{(j)}\right)f$ and $\left(\sum_{\nu=1}^{\infty}K_\nu^{(j)}\right)g$ both are elements in $\mathfrak{M}^{(j)}$, $\sum_{k=1}^{\infty}|a_k|^2<\infty$ and $\sum_{k=1}^{\infty}|b_k|^2<\infty$. Moreover, since by hypotheses $\varphi(\lambda)$ is regular in $\overline{D}$, there exists a finite positive number M such that $|\varphi^{(\alpha-1)}(\lambda)|\leqq M$, $\alpha=1,2,\cdots,m_j$, in $\overline{D}$. By virtue of Schwarz's inequality we have therefore

$$\begin{aligned}\sum_{\nu=1}^{\infty}|(K_{\nu}^{(j)}f, g)|\,|\varphi^{(\alpha-1)}(z_{\nu}^{(j)})| &= \sum_{\nu=1}^{\infty}|(K_{\nu}^{(j)}f, K_{\nu}^{(j)}g)|\,|\varphi^{(\alpha-1)}(z_{\nu}^{(j)})| \\ &\leqq \sum_{\nu=1}^{\infty}\|K_{\nu}^{(j)}f\|\,\|K_{\nu}^{(j)}g\|\,M \\ &\leqq \frac{M}{2}\sum_{\nu=1}^{\infty}(\|K_{\nu}^{(j)}f\|^2+\|K_{\nu}^{(j)}g\|^2), \\ &\leqq \frac{M}{2}\left(\sum_{k=1}^{\infty}|a_k|^2+\sum_{k=1}^{\infty}|b_k|^2\right)<\infty.\end{aligned}$$

Thus the theorem has been proved.

Corollary 1. Let N be a bounded normal operator, let the accumulation points of its point spectrum form a denumerably infinite set, let D be a domain whose boundary ∂D is a rectifiable closed Jordan curve belonging to the resolvent set of N and contains completely the point spectrum $\{z_\nu\}$ of N and its accumulation points inside itself, let K_ν be the eigenprojector of N corresponding to the eigenvalue z_ν, and let g_λ be the solution of the equation $\lambda x-Nx=f$ where $f\in\mathfrak{D}((\lambda I-N)^{-1})$. If ∂D does not contain inside itself all points of the continuous spectrum of N except its subset as a set of accumulation points of $\{z_\nu\}$, then for every $g\in\mathfrak{H}$

$$\frac{1}{2\pi i}\int_{\partial D}(g_\lambda, g)d\lambda=\sum_{\nu=1}^{\infty}(K_\nu f, g),$$

where the line integration is taken in the counterclockwise direction; and moreover the series of the right-hand side converges absolutely.

Proof. This is a direct consequence of Theorem A.

Remark. It is to be noted that the function (g_λ, g) of a complex variable λ has denumerably infinite non-isolated essential singularities inside ∂D.

Corollary 2. Let N be a bounded normal operator in $\mathfrak{H}$, let D be a domain whose boundary ∂D is positively oriented and satisfies all the conditions and the assumption stated in Corollary 1, and let $\{\varphi_\mu\}$ be a complete orthonormal set in $\mathfrak{H}$. If

$$(2)\qquad \frac{1}{2\pi i}\int_{\partial D}\lambda((\lambda I-N)^{-1}\cdot\varphi_\mu, \varphi_\mu)d\lambda$$

does not vanish, the numerical value of this integral gives an eigenvalue of N and φ_μ is a normalized eigenelement of N corresponding to that eigenvalue. (2) here means, however,

$$\left(\frac{1}{2\pi i}\int_{\partial D}\lambda(\lambda I-N)^{-1}d\lambda\cdot\varphi_\mu, \varphi_\mu\right)$$

different from the original meaning of the left member of (1).

Proof. Let z_ν be an arbitrary eigenvalue of N, K_ν the corresponding eigenprojector of N, and f_ν an arbitrary eigenelement of N corresponding to the eigenvalue z_ν. Since f_ν is given by a linear combination of elements belonging to $\{\varphi_\mu\}$, we can and do write f_ν

$=\sum_{k\geq 1} a_k\varphi_{n_k}$, $\varphi_{n_k}\in\{\varphi_\mu\}$. Then, by making use of the relation $K_\nu f_\nu = f_\nu$ and Parseval's formula, we can find without difficulty the relation $K_\nu\varphi_{n_k}=\varphi_{n_k}$. This result shows that there exists necessarily an element of $\{\varphi_\mu\}$ as a normalized eigenelement of N for each of the eigenvalues. Since it is easily verified that the result of Theorem A is also applicable in our case, we obtain

$$\frac{1}{2\pi i}\int_{\partial D}\lambda((\lambda I-N)^{-1}\cdot\varphi_\mu,\varphi_\mu)d\lambda=\sum_\nu z_\nu(K_\nu\varphi_\mu,\varphi_\mu),$$

where $\sum_\nu$ denotes the sum for all eigenvalues of N. We find therefore that the integral (2) never vanishes when and only when φ_μ is a normalized eigenelement corresponding to a non-zero eigenvalue of N and that the non-zero numerical value of (2) gives the eigenvalue of N corresponding to the eigenelement φ_μ.

Corollary 3. Let N be a bounded normal operator in $\mathfrak{H}$, let D be a domain whose boundary ∂D is a circle with center at the origin and radius $R>||N||$ and belongs to the resolvent set of N, and let f be an eigenelement of N. Then the eigenvalue of N corresponding to the eigenelement f is given by

$$\frac{1}{2\pi i||f||^2}\int_{\partial D}\lambda((\lambda I-N)^{-1}\cdot f,f)d\lambda,$$

where ∂D is positively oriented.

Proof. Let $\Delta(N)$ be the continuous spectrum of N, $\{K(z)\}$ the complex spectral family associated with N, and K_ν, $\{z_\nu\}$ the same symbols as those defined in Corollary 1. Since it is easily verified by hypotheses that R is greater than the spectral radius of N, and since, hence, ∂D has $\{z_\nu\}$ and $\Delta(N)$ inside itself, by reasoning like that used to prove (1) we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial D}\lambda((\lambda I-N)^{-1}f,f)d\lambda&=\sum_\nu z_\nu||K_\nu f||^2+\frac{1}{2\pi i}\int_{\partial D}\int_{G}\frac{\lambda}{\lambda-z}d||K(z)f||^2d\lambda\\&=\sum_\nu z_\nu||K_\nu f||^2+\frac{1}{2\pi i}\int_{\Delta(N)}\int_{\partial D}\left(1+\frac{z}{\lambda-z}\right)d\lambda d||K(z)f||^2\\&=\sum_\nu z_\nu||K_\nu f||^2+\int_{\Delta(N)}zd||K(z)f||^2,\end{aligned}$$

where $\sum_\nu$ denotes the sum for all eigenvalues of N. In addition to it, by the definition on f, clearly $K(\delta)f$ vanishes for every subset δ of $\Delta(N)$ and there exists a unique eigenvalue $z_\alpha\in\{z_\nu\}$ such $\sum_\nu z_\nu||K_\nu f||^2 =z_\alpha||f||^2$.

In consequence, we have the desired relation

$$\frac{1}{2\pi i||f||^2}\int_{\partial D}\lambda((\lambda I-N)^{-1}\cdot f,f)d\lambda=z_\alpha.$$