132. On $L^{(k)}$-Transform and the Generalized Laplace Transform

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1.*) Let $f(\zeta)$ be the Laplace transform of the function $F(x)$:

$$f(\zeta)=L(F)=\int_0^\infty e^{-\zeta x}F(x)dx.$$

If this Laplace integral $L(F)$ is convergent for a complex number ζ, then this means that the function

$$\Phi(x)=\int_0^x e^{-\zeta x}F(x)dx=(e^{-\zeta x}F(x))*1$$

has a limit for $x\to\infty$. Here $g(x)*h(x)$ means

$$\int_0^x h(\tau)\,g(x-\tau)\,d\tau=\int_0^x h(x-\tau)\,g(\tau)\,d\tau.$$

If $L(F)$ is not convergent, then we consider the Cesàro's k th order (C, k) mean of Φ:

$$m_k(x)=\frac{k}{x^k}\{(e^{-\zeta x}F)*1*x^{k-1}\}=\frac{(e^{-\zeta x}F)*x^k}{x^k}$$

where k is a positive integer, and for $k=0$ we put $m_0(x)=\Phi(x)$.

If this mean has a limit for $x\to\infty$, then we say that the Laplace integral is (C, k) *convergent* for ζ, or for the values ζ *the* $L^{(k)}$*-transform of* F:

$$L^{(k)}(F)=\lim_{x\to\infty}\frac{(e^{-\zeta x}F)*x^k}{x^k} \text{ exists.}$$

The domain of convergence of $L^{(k)}(F)$ is a half plane: $\{\zeta\,|\,Re(\zeta)>\beta_k\}$ for some real number $\beta_k(-\infty\leq\beta_k\leq+\infty)$.

For any pair of positive integers k and k', such that $k'>k$, the inequality $\beta_{k'}\leqq\beta_k$ follows. So, for $k\to\infty$, β_k converges to the limit B. (B can be finite or $\pm\infty$.)

It can appear that $L^{(k)}(F)$ is convergent for the first time for some (large) k, whereas for the other (smaller) k', $L^{(k')}(F)$ do not converge.

The function $L^{(k)}(F)$ is analytic in the interior of the convergent domain $\{\zeta\,|\,Re(\zeta)>\beta_k\}$ and coincides with the functions $L^{(k')}(F)$ for $k'>k$, in the half plane defined by $\{\zeta\,|\,Re(\zeta)>\beta_k\}$. The totality of $L^{(k)}(F)$ for $k\geq 0$ defines a function $f(\zeta)$ in the half plane $\{\zeta\,|\,Re(\zeta)>B\}$ which we call L^∞-transform of F.

Using L^∞-transform, thus, we can examine the analytic continuation of $L(F)$ in the domain outside the axis of convergence of

*) See reference G. Doetsch [1]. In **1** our notations conform to those of G. Doetsch.

Lapalce integral.

Let x_0 be the infinimum of real numbers x such that the analytic continuation of $f(\zeta)=L(F)$ is holomorph for $\{\zeta \mid Re(\zeta)>x\}$. Then we call the half plane $Re(\zeta)>x_0$ *the holomorph half plane of* $L(F)$. According to G. Doetsch, it is an unsolved problem whether the holomorph half plane of $L(F)$ coincides with the region of convergence of $L^{\infty}(F)$ or not. Therefore at the present stage of our knowledge the method using $L^{(k)}$-transform is one of the best possible way to obtain the holomorph half plane of $L(F)$.

Now, besides these classical methods, there is another way to obtain the holomorph half plane of $L(F)$. In the preceding papers [2], [3], [4] we considered Laplace transform

$$f(\zeta)=L(F)=\int_0^{\infty} e^{-x\zeta} F(x)dx$$

as the functional on the space $\Phi(\xi, \eta)$, or mapping from ξ-space to $Z'(\eta)$ space. Using such a view, the generalized Laplace transform $\mathfrak{L}(F)$ are defined on the whole ζ plane.

In this paper, we show that $L^{\infty}(F)$ coincides with the generalized Laplace transform $\mathfrak{L}(F)$ in the region of convergence, so that the way using functional analysis to examine analytic continuation may also be considered as one of the best possible way at present. Further, we remark that our method has the advantage of finding directly the analytic continuation, while L^{∞}-transform generally requires infinite steps of mean.

2. We denote $\mathfrak{D}$ the space of C^{∞} functions (of one real variable) having compact carriers. Its topology is that given in L. Schwartz [5]. We denote $Z(\eta)$ the space of functions $\psi(\eta)$ of Fourier transforms of $\mathfrak{D}$, the notion of which is introduced and is investigated by E. M. Gelfand and G. E. Sylov [7], [8], [9] and by L. Ehrenpreis [10].

Its topology is given by the neighbourhoods system such that whose neighbourhood consists of the Fourier transforms of the functions belonging to a neighbourhood of $\mathfrak{D}$.

Further we denote $\mathfrak{D}'_+$ [6] the space of distributions whose carriers are contained in $x\geq 0$. We denote $\zeta=\xi+i\eta$, and consider the Laplace tronsforms $\int_0^{\infty} e^{-\zeta x}F(x)\,dx$.

For any fixed ξ and for any distribution $F\in\mathfrak{D}'_+$, $\exp(-\xi x)F\in\mathfrak{D}'_+$. Its Fourier transform $\mathfrak{F}(\exp(-\xi x)F)$ is a functional on the space $Z(\eta)$. We call the generalized Laplace transform this mapping: $\xi\to\mathfrak{F}(\exp(-\xi x)F)$ from thes pace of the real number $R(-\infty<\xi<+\infty)$ to the space $Z'(\eta)$ and denote it symbolically by

$\alpha(F)=\int_0^{\infty}\exp(-\zeta x)F(x)\,dx$ [4].

Now we restrict our considerations to the case that the distribution F is an ordinary function. Let $F(x)$ be a real or complex valued function which is defined in the interval $0\leq x$ and is locally summable and locally finite there. Then $F(x)$ is a distribution which belongs to $\mathfrak{D}'_{+}$. Further, for simplicity, we assume that $F(x)$ satisfies the following conditions (as stated in 1):

(i) The domain of convergence of the Laplace integral $L(F)=\int_0^\infty e^{-\zeta x}F(x)\,dx$ is a half plane $\{\zeta\,|\,Re(\zeta)>\beta_0\}$ where $\beta_0<+\infty$.

(ii) The L^∞-transform converges on the half plane $\{\zeta\,|\,Re(\zeta)>B\}$.

(iii) We assume $B<\beta_0$.

We can see that the following theorem holds.

Theorem. *If $L^\infty(F)$ converges in the rigion $Re(\zeta)>B$, then in the same region, the generalized Laplace transform coincides with $L^\infty(F)$ and represents the analytic continuation of the ordinary Laplace transform.*

Proof. For any point ζ_0 such that $Re(\zeta_0)>B$, there exists positive integer k such that the interior of domain of convergence of $L^{(k)}(F)$ contains ζ_0, i.e., $Re(\zeta_0)>\beta_k$. We assume also for a point $\zeta_0'(Re(\zeta_0')>Re(\zeta_0))$ the ordinarly Laplace transform converges.

Since the integral $\int_{-\infty}^{+\infty} e^{-i\eta\tau}\psi(\eta)\,d\eta$ converges uniformely for $\tau(0\leq\tau\leq x)$ for any fixed element $\psi(\eta)\in Z(\eta)$, we see that

$$\frac{1}{x^k}\int_0^x e^{-\xi\tau}<e^{-i\eta\tau},\ \psi(\eta)>_\eta F(\tau)\,(x-\tau)^k\,d\tau=<m_k(x),\ \psi(\eta)>_\eta$$

by the change of order of integration. So, for any fixed ξ and x, $m_k(x)=\dfrac{(e^{-\zeta x}F)*x^k}{x^k}$ defines a functional on $Z(\eta)$. Since $<e^{-i\eta\tau},\psi(\eta)>_\eta=\varphi(\tau)$ belongs $\mathfrak{D}(\tau)$, $\varphi(\tau)$ has a compact carrier $[x_1\leq\tau\leq x_0]$, and

$$\lim_{x\to\infty}\frac{1}{x^k}\int_0^\infty e^{-\xi\tau}<e^{-i\eta v},\ \psi(\eta)>_\eta F(\tau)\,(x-\tau)^k\,d\tau$$

always exists and equals

$$\frac{1}{x_0^k}\int_0^{x_0} e^{-\xi\tau}<e^{-i\eta\tau},\ \psi(\eta)>_\eta F(\tau)\,(x_0-\tau)^k\,d\tau.$$

In the space Z' a weakly convergent series coincides with strongly convergent series similarly in D'.

So, for any ξ, $\lim_{x\to\infty} m_k(x)$ defines a functional $\mathfrak{L}^k(F)\in Z'(\eta)$. On the other hand, for any $\xi>\beta_k$, $m_k(x)$ converges to the function $L^k(F)$ for $x\to\infty$ in the ordinary sense. Since the analytic function satisfies $L^k(F)=o(|\eta|^{k+1})$ for $\eta\to\infty$([1], p. 333) for $\xi>\beta_k$, $L^k(F)\in S'(\eta)\subset Z'(\eta)$. So, we see that $L^k(F)$ coincides with $\mathfrak{L}^k(F)$ for $\{\zeta\,|\,Re(\zeta)>\beta_k\}$.

Since $\varphi(\tau)\in\mathfrak{D}(\tau)$, $\varphi(x)\,e^{-\xi x}\,F(x)*1$ converges in the ordinary sense with $x\to\infty$. Therefore by virtue of the property of Cesàro's mean,

$$\lim_{x\to\infty}(\varphi(x)\,e^{-\xi x}\,F(x)*1)=\lim_{x\to\infty}\frac{\varphi(x)\,e^{-\xi x}\,F(x)*x^k}{x^k}$$

$$=\lim_{x\to\infty}<\frac{e^{-\zeta x}\,F(x)*x^k}{x^k},\,\psi(\eta)>_\eta=\lim_{x\to\infty}<m_k(x),\,\psi(\eta)>_\eta$$

$$=<\mathfrak{L}^k(F),\,\psi(\eta)>=<L^k(F),\,\psi(\eta)>.$$

The first term means the inner product between the generalized Laplace transform $\mathfrak{L}(F)$ and $\psi(\eta)$. Therefore, we can see the following relation $L^{(k)}(F)=\mathfrak{L}(F)$, and also the analyticity of $\mathfrak{L}(F)$ in the domain of convergence $L^{(k)}(F)$. Since we can take arbitrary large k, we can see the analyticity of $\mathfrak{L}(F)$ on the half plane $\{\zeta\,|\,Re(\zeta)>B\}$.

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