131. On Generalized Laplace Transforms

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1. To investigate Laplace transforms of functionals is important relating two parts of analysis. On one hand it contributes to the developments of the theory of functional analysis and its applications, for instance, to the theory of partial defferential equations (Leray [1]). On the other hand it will contribute also to the investigations of the classical analysis, especially of the classical theory of Laplace transforms.

However, it seems to us that the systematic applications of functional analysis to the developments of the classical theory of Laplace transforms are still few at present.

Laplace transforms of distributions are investigated in detail by L. Schwartz [2]. However, he limited his considerations about Laplace transforms of the distribution T to the case such that $e^{-x\xi}T\in S'$. Such a limitation causes some confinements for the development and applications of the theory.

On the other hand, Fourier transforms of general distributions ($\in\mathfrak{D}'$) (not of tempered distributions $\in S'$) are investigated by E. M. Gelfand and G. E. Sylov [3], [4], [5] and L. Ehrenpreis [6]. These investigations, however, concern mainly to Fourier transforms, and the systematic theory of Laplace transforms is also not discussed.

In the preceding papers [7], [8], we considered divergent integrals $\int_0^\infty e^{i(\sigma+i\tau)s}v(\sigma,\tau,s)ds$ as functionals $\in\Phi'$. In case v is indepent of σ,τ, these integrals are Laplace integrals. Examples cited there are also of Laplace integrals. But the details of the theory were not discussed.

In this and the following papers we will consider the systematic theory of generalized Laplace integrals and its applications. In the preceding papers we used the dual of the space of the tensor product $Z(\sigma)\otimes D(\tau)$. But in this and following papers we will use mapping $\tau\rightarrow Z'(\sigma)$, though they are not essentially so different.

2. Let X^n, Y^n, Ξ^n, H^n be the n-dimensional real vector spaces, and Ξ^n+iH^n, X^n+iY^n be the n-dimensional complex vector spaces. We denote x, y, ξ, η, an element of X^n, Y^n, Ξ^n, H^n each, and call $\zeta=\xi+i\eta$, $z=x+iy$. To simplify notations, we use usually abbreviated writing as following: ηy means $\eta_1y_1+\cdots+\eta_ny_n$, $y\geqq 0$ means $y_1\geq 0,\cdots,$ $y_n\geq 0$ where $\eta=(\eta_1,\cdots,\eta_n)$ and $y=(y_1,\cdots y_n)$.

Let $\mathfrak{D}$ denote the space of C^∞ functions of compact carriers, defined on Y^n. Its topology is that given in L. Schwartz [9]. We denote $Z(\eta)$ the space of functions $\psi(\eta)$ of Fourier transforms of functions of $\mathfrak{D}$, the notion of which is introduced and investigated by E. M. Gelfand and G. E. Sylov [3], [4], [5] and L. Ehrenpreis [6]. The topology of $Z(\eta)$ is given by the neighborhoods system such that whose neighborhood consists of the Fourier transforms of the functions belonging to a neighborhood of $\mathfrak{D}$. Further we denote $\mathfrak{D}'(Y)$ the space of distributions defined on Y^n, $\mathfrak{D}'_+(Y)$ the space of distributions whose carriers are contained in $y\geq 0$, $\mathfrak{D}'_-(Y)$ the space of distributions whose carriers are contained in $y\leq 0$.

We consider the Laplace transform

$$\int_0^\infty e^{-\zeta y}\,T\,dy,\quad \int_{-\infty}^0 e^{-\zeta y}\,T\,dy,\quad \int_{-\infty}^{+\infty} e^{-\zeta y}\,T\,dy$$

for any distribution $T\in\mathfrak{D}'_+, \mathfrak{D}'_-$ or $\mathfrak{D}'$.

For any fixed ξ, for any distribution T, $\exp(-\xi y)\,T\in\mathfrak{D}'_+, \mathfrak{D}'_-$ or $\mathfrak{D}'$. Its Fourier transform $\mathfrak{F}(\exp(-\xi y)T)$ is a functional on the space $Z(\eta)$.

Definition 1. We call *the generalized Laplace transform* of T $(\in\mathfrak{D}')$ these mappings: $\xi\to\mathfrak{F}(\exp(-\xi y)T)$ from the space of the n-dimensional real vector space Ξ^n to the space $Z'(\eta)$, and denote it symbolically by $\mathfrak{L}(T)$ or

$$\int_0^\infty \exp(-\zeta y)\,T\,dy,\quad \int_{-\infty}^0 \exp(-\zeta y)\,T\,dy \ \text{ or } \int_{-\infty}^{+\infty} \exp(-\zeta y)\,T\,dy \ \text{ each.}$$

3. We see here some important properties of the generalized Laplace transforms.

First we cite some properties of the space Z and Z'.

Lemma 1. *Let S denote the space of rapidly decreasing C^∞ functions. The space $\mathfrak{D}$, Z is dence in the space S in the topology of S.*

Lemma 2. *Both the spaces $\mathfrak{D}$ and Z are complete, Montel and reflexive.*

Lemma 3. *In the space Z' a weakly convergent series coincides with strongly convergent series similarly in $\mathfrak{D}'$.*

Proof. Each of these properties is deduced from the properties of $\mathfrak{D}, \mathfrak{D}'$ by homeomorph mappings (Fourier transforms) to Z, or to Z' each.

Now we turn to the notion of the generalized Laplace transforms.

Toeorem 1. *If the Laplace integral for locally summable and locally finite function $F(y)$: $L(F)=\int_0^\infty e^{-\zeta y}F(y)\,dy$ converges (in the ordinary sense) for $\xi>\beta$, then $L(F)$ coincides with the generalized Laplace integral $\mathfrak{L}(F)$ for ξ.*

Proof. Since the integral $\int_{-\infty}^{+\infty} e^{-i\eta y}\psi(\eta)\,d\eta$ converges uniformly for $y \geq 0$ for any fixed element $\psi(\eta) \in Z(\eta)$, we see that

$$\int_0^y e^{-\xi y} < e^{-i\eta y}, \psi(\eta) >_\eta F(y)dy = < m_y(\eta), \psi(\eta) >_\eta$$

by the change of orders of integration where $m_y(\eta) = \int_0^y e^{-\xi y} F(y)dy$.

So for any fixed ξ and $y, m_y(\eta)$ defines a functional on $Z(\eta)$. Since $< e^{-i\eta y}, \psi(\eta) >_\eta = \varphi(y)$ belongs to $D(y)$, $\varphi(y)$ has a compact carrier.

So $\lim_{y\to\infty} \int_0^y e^{-\xi y} < e^{-i\eta y}, \psi(\eta) >_\eta F(y)dy$ exists, i.e., $m_y(\eta)$ is weakly convergent to $\mathfrak{F}(e^{-\xi y}F) = \mathfrak{L}(F)$ for $y \to \infty$ in the space $Z'(\eta)$. Since in the space Z', a weakly convergent series coincides with strongly convergent series similary in $\mathfrak{D}'$, $\lim_{y\to\infty} m_y(\eta)$ equals a functional $\mathfrak{L}(F) \in Z'(\eta)$ for any ξ.

On the other hand for any $\xi > \beta$, $m_y(\eta)$ converges in the ordinary sense to the Laplace transformed function $L(F)$.

Since the analytic function $L(F)$ satisfies $L(F) = o(|\eta|)$ for $|\eta| \to \infty$, for $\xi > \beta$ [11], $L(F)$ belongs to the space $S'(\eta)$ which is the subspace of $Z'(\eta)$. So we see that $L(F)$ coincides with $\mathfrak{L}(F)$ for $\{\zeta \mid Re(\zeta) > \beta\}$.

Theorem 2. *The generalized Laplace transforms of $\mathfrak{D}'$ are analytic functions of ξ on the whole space E^n with values in $Z'(\eta)$.*

Proof. We prove here for $\mathfrak{D}'_+$. Similar proof is verified for the other cases too. We put $\omega = \xi + i\sigma$, $< e^{-i\eta y}, \psi(\eta) >_\eta = \varphi(y)$ where $\psi(\eta) \in Z(\eta)$. Then

$$\begin{aligned}
&\lim_{\omega\to\omega_0} \frac{1}{\omega-\omega_0}\left\{ < \int_0^\infty e^{-(\omega+i\eta)y}\, T\, dy, \psi(\eta) >_\eta - < \int_0^\infty e^{-(\omega_0+i\eta)y}\, T\, dy, \psi(\eta) >_\eta \right\} \\
&\qquad = \lim_{\omega\to\omega_0} \frac{1}{\omega-\omega_0} < (e^{-\omega y} - e^{-\omega_0 y})\, T, \varphi(y) >_y \\
&\qquad = \lim_{\omega\to\omega_0} \frac{1}{\omega-\omega_0} < T, (e^{-\omega y} - e^{-\omega_0 y})\varphi(y) >_y \\
&\qquad = < T, -ye^{-\omega_0 y}\varphi(y) >_y = < -ye^{-\omega_0 y}\, T, \varphi(y) >_y \\
&\qquad = < \int_0^\infty -x\, e^{-(\xi_0+i\sigma_0+i\eta)x}\, T\, dx, \psi(\eta) >_\eta .
\end{aligned}$$

This weak convergence can be replaced by the strong convergence by virtue of Lemma 3. So we see that Theorem 2 holds.

In 2 we considered the generalized Laplace transform of arbitrary distribution $T (T \in \mathfrak{D}'_+$ or $\mathfrak{D}'_-$, or $\mathfrak{D}')$. We can introduce similarly the generalized Laplace transforms for functional $S(\eta) \in Z'(\eta)$ such that $e^{-\eta x}S(\eta) \in Z'(\eta)$.

For any fixed x and for any functional $S \in Z'(\eta)$ such that $e^{-\eta x}S \in Z'(\eta)$, its Fourier transform $\mathfrak{F}(\exp(-\eta x)\, S)$ is a distribution $\in D'$.

We call also the generalized Laplace transform this mapping: $x \to \mathfrak{F}(\exp(-\eta x)\, S)$ from the space of the n-dimensional real vector space X^n to the space $\mathfrak{D}'(Y)$ and denote it symbolically by

$$\int_{-\infty}^{+\infty} \exp(-z\eta)\, S\, d\eta.$$

Similarly to Theorem 2, we can see the following theorem.

Theorem 3. *The generalized Laplace transforms of Z' are analytic functions of x on the defined region of X^n with values in $\mathfrak{D}'(y)$.*

Proof. We can prove this theorem quite similarly to the proof of Theorem 2.

The generalized Laplace transform of $Z'(\eta)$ will be not so useful as the tarnsform of $\mathfrak{D}'(y)$, because the confinement $e^{-\eta x} S \in Z'(\eta)$ is necessary for S, though this confinement is less restrictive than $e^{-\eta y} s \in S'(\eta)$ [2].

4. We shall see here some properties of operations about the generalized Laplace transforms.

The inverse formula is given by Fourier transform.

Let S be a Laplace transform of $T \in \mathfrak{D}'$, i.e.,

$$S(\xi, \eta) = \int_0^{\infty} e^{-\zeta y}\, T\, dy, \text{ or} = \int_{-\infty}^{0} e^{-\zeta y}\, T\, dy, \text{ or} = \int_{-\infty}^{+\infty} e^{-\zeta y}\, T\, dy.$$

Here $T \in D'_+$, or D'_-, or D' each, and $S(\xi, \eta)$ means the map from the point ξ to the space $Z'(\eta)$.

Then it follows that $e^{-\xi y}\, T(y) = \dfrac{1}{(2\pi)^n} \displaystyle\int_{-\infty}^{+\infty} e^{i\eta y}\, S(\xi, \eta) d\eta$, where expression means inverse Fourier transform from $Z'(\eta)$ to $\mathfrak{D}'(y)$. This expression can be written also as follows:

$$T(y) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} e^{(\xi + i\eta) y}\, S(\xi, \eta) d\eta$$

for any $\xi \in \Xi^n$. For $S(\xi, \eta) = \mathfrak{L}(T)$, we denote this formula by $T = \mathfrak{L}^{-1}(S)$.

We will see next some properties of convolution and multiplication operations with the generalized Laplace transforms. First we introduce some notions conforming to E. M. Gelfand and G. E. Sylov [3] [4] [5].

Definition 2. We shall use the term *basic space* for any complete linear topological space which is an (algebraic) linear subspace of S and whose topology is finer than that of S. We denote it by Φ and its dual by Φ'. The space $Z, \mathfrak{D}, S$ is one of the basic spaces.

Definition 3. We call a function f is *a multiplier* in the space Φ' if f enjoys the properties (i) $\phi \in \Phi'$ implies $f \cdot \phi \in \Phi'$, and further (ii) $\phi_n \to 0$ implies $f \cdot \phi_n \to 0$ where $\phi_n \in \Phi'$.

Definition 4. We call a functional $f_0 \in \Phi'$ is *a convolutioner* of the space Φ, if f_0 enjoys the following properties:

(i) For any $\varphi\in\Phi, f_0*\varphi\equiv<f_0(\eta), \varphi(y+\eta)>_\eta=\psi(y)\in\Phi$,

(ii) $\varphi_\nu\to 0$ implies $f_0*\varphi_\nu\to 0$ in the topology of Φ.

We cite here the following lemma by E. M. Gelfand and G. E. Sylov [5] without proof.

Lemma 4. *Let Φ be a translation inveriant basic space in which translation operation is continuous. Call Ψ its Fourier transform: $\Psi=\mathfrak{F}(\Phi)$. If a function g is a multiplier in the space Ψ' then $f=\mathfrak{F}^{-1}(g)$ is a convolutioner in the space Φ' and the following formula holds for any element $f_1\in\Phi'$: $\mathfrak{F}(f*f_1)=\mathfrak{F}(f)\mathfrak{F}(f_1)$.*

Now we define operations of multiplication and convolution between the generalized Laplace transforms.

Definition 5. Let $f(\xi,\eta)$ be a function on Ξ^n+iH^n or a function type generalized Laplace transform, and let $S(\xi,\eta)$ be a generalized Laplace transform. If for any fixed ξ, the multiplication $f(\xi,\eta)\,S(\xi,\eta)$ in the $Z'(\eta)$ space is possible, and the multiple belongs to $Z'(\eta)$, then we call *the multiplication* between $f(\xi,\eta)$ and $S(\xi,\eta)$ is possible, and denote this correspondence $\xi\to f(\xi,\eta)\,S(\xi,\eta)$ $(\Xi^n\to Z'(\eta))$ simply by $f(\xi,\eta)\,S(\xi,\eta)$.

Similarly we define convolution operation between the generalized Laplace transforms.

Definition 6. Let $S_1(\xi,\eta)$ and $S_2(\xi,\eta)$ be two generalized Laplace transforms such that for any fixed ξ the convolution between two elements of $Z'(\eta)$: $S_1(\xi,\eta)\underset{\eta}{*}S_2(\xi,\eta)$ is possible. Then we say that *the convolution* between S_1 and S_2 is possible, and denote the mapping $\xi\to S_1(\xi,\eta)\underset{\eta}{*}S_2(\xi,\eta)$ $(\Xi^n\to Z'(\eta))$ simply by $S_1(\xi,\eta)\underset{\eta}{*}S_2(\xi,\eta)$.

Now we can see the following theorem holds.

Theorem 4. *Suppose that $f(\xi,\eta)=\mathfrak{L}(g)$ be a Laplace transformed function of $g\in\mathfrak{D}'(y)$. If $f(\xi,\eta)$ is a multiplier in the space $Z'(\eta)$ for any ξ, then its inverse Laplace transform $g(y)=\mathfrak{L}^{-1}(f)$ is a convolutioner in the space $\mathfrak{D}'(y)$, and satisfies the following equality for any element $g_1\in\mathfrak{D}'(y)$: $\mathfrak{L}(g*g_1)=\mathfrak{L}(g)\cdot\mathfrak{L}(g_1)$.*

Proof. Taking the space $\mathfrak{D}(y)$ as the space Φ, $f(\xi,\eta)$ in this theorem as g, we apply Lemma 4. Since $f(\xi,\eta)$ is a multiplier in the space $Z'(\eta)$ for any ξ,

$$e^{-y\xi}g(y)=\frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty}e^{i\eta y}f(\xi,\eta)\,d\eta$$

is a convolutioner in the space $\mathfrak{D}'(y)$ for any ξ. By the conclusion of Lemma 4, we see $\mathfrak{F}(e^{-y\xi}\,g(y)\underset{y}{*}e^{-y\xi}\,g_1(y))=\mathfrak{F}(e^{-y\xi}\,g(y)\cdot\mathfrak{F}(e^{-y\xi}\,g_1(y))$.

Since $e^{-y\xi}\,g(y)*e^{-y\xi}\,g_1(y)=e^{-y\xi}\{g(y)*g_1(y)\}$ we can see the theorem holds.

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